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In memory of Viktor Scharnitzky, mathematician, college professor

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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance. The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve the following systems of equations over the ordered pairs of real numbers.

a) $(x+1)y = 0$, $(x^2-1)(y+1) = 0$ b) $x^2 + xy = 2$, $y^2 - 2xy = 5$

Solution: a) From the first equation $y = 0$ or $x = -1$; from the second $y = -1$ or $x^2 = 1$. If $y = 0$, then $x^2 = 1$, so $(1, 0)$ and $(-1, 0)$ are solutions. If $x = -1$, then the second equation holds for every y . So the solutions are $(1, 0)$ and $(-1, t)$, where t is any real number.

b) Multiplying the first equation by 5 and the second by 2, and subtracting: $5x^2 + 9xy - 2y^2 = 0$, that is, $(5x - y)(x + 2y) = 0$.

If $y = 5x$: $6x^2 = 2$, so $\left(\frac{1}{\sqrt{3}}, \frac{5}{\sqrt{3}}\right)$ and $\left(-\frac{1}{\sqrt{3}}, -\frac{5}{\sqrt{3}}\right)$.

If $x = -2y$: $4y^2 - 2y^2 = 2$, so $(-2, 1)$ and $(2, -1)$.

All four pairs satisfy the system. (The KoMaL solution does not discuss part b); this solution is ours.)

2. Let $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = 2x^2 - 2x + 4$. Prove the identity $f(k+2) = f(k) + 8k + 4$.

Solution: $f(k+2) = 2(k+2)^2 - 2(k+2) + 4 = 2k^2 + 6k + 8$, and $f(k) + 8k + 4 = 2k^2 - 2k + 4 + 8k + 4 = 2k^2 + 6k + 8$.

3. Find the values of the parameter a for which one root of the following equations is twice the other root.

1) $x^2 + 3ax + 2a^2 = 0$ 2) $x^2 - 3ax + 2a^2 - 1 = 0$

3) $x^2 - 3(a+1)x + 2a^2 + 6a = 0$ 4) $x^2 + ax - 2a - 4 = 0$

Solution: Let the roots be x_1 and $x_2 = 2x_1$, and use Viète's formulas.

1) $3x_1 = -3a$, $2x_1^2 = 2a^2$: this holds for every a , since the roots are $-a$ and $-2a$.

2) $3x_1 = 3a$, $2x_1^2 = 2a^2 - 1$, so $2a^2 = 2a^2 - 1$, which is impossible: there is no such a .

3) $3x_1 = 3(a+1)$, $2x_1^2 = 2a^2 + 6a$, so $2(a+1)^2 = 2a^2 + 6a$, and $a = 1$. Then the equation is $x^2 - 6x + 8 = 0$, with roots 2 and 4.

4) $D = a^2 + 4(2a+4) = (a+4)^2$, so the roots are 2 and $-a-2$. If $-a-2 = 4$, then $a = -6$; if $-a-2 = 1$, then $a = -3$. So $a = -6$ or $a = -3$.

4. In the coordinate plane, e and f are two parallel lines, and k is a circle between them. Reflecting k in the line e we get the circle $k_1: x^2 + y^2 - 4x - 12y + 39 = 0$, and reflecting k in the line f we get the circle $k_2: x^2 + y^2 - 16x + 12y + 99 = 0$. Find the point with respect to which k_1 and k_2 are centrally symmetric, and the equation of the line with respect to which k_1 and k_2 are symmetric. Find the distance between the lines e and f .

Solution: k_1 is $(x - 2)^2 + (y - 6)^2 = 1$, with center $C_1(2, 6)$, and k_2 is $(x - 8)^2 + (y + 6)^2 = 1$, with center $C_2(8, -6)$. The center of symmetry is the midpoint $F(5, 0)$ of C_1C_2 , and the axis of symmetry is the perpendicular bisector of C_1C_2 , which is $x - 2y = 5$. If K is the center of k , then C_1 and C_2 are its reflections in e and f , and since K is between the lines, C_1C_2 is twice the distance of the lines. So the distance of e and f is $\frac{C_1C_2}{2} = \frac{\sqrt{180}}{2} = \sqrt{45} = 3\sqrt{5}$ units.

5. The base of a pyramid with four lateral faces is the rhombus $ABCD$. The perpendicular projection of the apex E onto the plane of the base is the intersection point F of the diagonals of the rhombus. Find the volume and the surface area of the pyramid, if the diagonals of the rhombus are $AC = 10$ cm and $BD = 18$ cm, and the height of the pyramid is $EF = 12$ cm.

Solution: The diagonals of the rhombus bisect each other at right angles, so from the right triangles AFE and BFE : $AE = CE = 13$ cm and $BE = DE = 15$ cm. The volume is

$$V = \frac{1}{3} \cdot \frac{10 \cdot 18}{2} \cdot 12 = 360 \text{ cm}^3$$

The side of the rhombus is $\sqrt{25 + 81} = \sqrt{106}$ cm. The lateral surface consists of four congruent triangles with sides 13 cm, 15 cm, and $\sqrt{106}$ cm. By Heron's formula, the area of one of them is $T_1 \approx 65.74 \text{ cm}^2$, so the surface area of the pyramid is

$$A \approx 90 + 4 \cdot 65.74 \approx 352.98 \text{ cm}^2$$

6. Solve the following inequalities over the real numbers.

$$\text{a) } \frac{\log_3 x}{\left(\log_{\frac{1}{3}} x + 3\right) \left(\log_{\frac{1}{3}} x - 1\right)} < 0 \quad \text{b) } \frac{\log_{\frac{1}{3}}^2 x + 2 \log_{\frac{1}{3}} x - 3}{\log_3 x} \geq 0$$

Solution: Let $t = \log_3 x$ ($x > 0$); then $\log_{\frac{1}{3}} x = -t$.

a) The inequality is $\frac{t}{(t - 3)(t + 1)} < 0$. The fraction is negative if $0 < t < 3$ or $t < -1$, that is, $1 < x < 27$ or $0 < x < \frac{1}{3}$.

b) The inequality is $\frac{t^2 - 2t - 3}{t} = \frac{(t - 3)(t + 1)}{t} \geq 0$, which holds if $-1 \leq t < 0$ or $t \geq 3$, that is, $\frac{1}{3} \leq x < 1$ or $x \geq 27$.

7. a) Prove that the sequence (a_n) is arithmetic if and only if $a_{n+1} = 2a_n - a_{n-1}$ for every natural number $n > 1$.
 b) The sequence (a_n) is arithmetic with difference d . There are real numbers p and q such that $a_{n+1} = 2pa_n - qa_{n-1}$ for every natural number $n > 1$. Find the possible values of p and q , if (a_n) is (i) not constant; (ii) constant with $a_1 \neq 0$; (iii) constant with $a_1 = 0$.

Solution: a) If the sequence is arithmetic with difference d , then $a_{n+1} - a_n = d = a_n - a_{n-1}$, so $a_{n+1} = 2a_n - a_{n-1}$. Conversely, if $a_{n+1} = 2a_n - a_{n-1}$ for every $n > 1$, then $a_{n+1} - a_n = a_n - a_{n-1}$, so the differences are all equal, and the sequence is arithmetic.

b) By part a), $a_{n+1} = 2a_n - a_{n-1}$, so $2pa_n - qa_{n-1} = 2a_n - a_{n-1}$, that is, $(2p - 2)a_n = (q - 1)a_{n-1}$ for every $n > 1$.

(i) If the sequence is not constant, then $d \neq 0$, and substituting $a_n = a_{n-1} + d$: $(2p - q - 1)a_{n-1} + (2p - 2)d = 0$ for every $n > 1$. Since a_{n-1} takes different values, both coefficients are 0: $2p - q - 1 = 0$ and $2p - 2 = 0$, so $p = 1$ and $q = 1$.

(ii) If every term is $a_1 \neq 0$, then $(2p - 2)a_1 = (q - 1)a_1$, so $q = 2p - 1$, where p is any real number.

(iii) If every term is 0, then p and q can be any real numbers.

8. From the point P of a circle we draw the chords $PA = 12$ cm and $PB = 16$ cm. The distance of the midpoint F of the chord PA from the line PB is $2\sqrt{3}$ cm. Find the radius of the circle.

Solution: F is 6 cm from P and $2\sqrt{3}$ cm from the line BP . Let $\angle BPA = \alpha$; then $\sin \alpha = \frac{2\sqrt{3}}{6} = \frac{\sqrt{3}}{3}$, so $\cos \alpha = \pm \sqrt{\frac{2}{3}}$, and there are two solutions. By the Law of Cosines $AB^2 = 12^2 + 16^2 - 2 \cdot 12 \cdot 16 \cos \alpha = 16(25 - 24 \cos \alpha)$, and by the Law of Sines $r = \frac{AB}{2 \sin \alpha}$. So

$$r_1 = 2\sqrt{\sqrt{3}(25\sqrt{3} - 24\sqrt{2})} \approx 8.05 \text{ cm}, \quad r_2 = 2\sqrt{\sqrt{3}(25\sqrt{3} + 24\sqrt{2})} \approx 23.13 \text{ cm}$$

(Note: the KoMaL solution gives the approximate values 4.03 and 11.57, which are half of the correct values; this is a slip.)