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In memory of Károly Kőváry, principal, mathematics teacher, my former colleague

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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance. The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve the following equations over the real numbers.

$$\begin{array}{ll} \text{a) } \sqrt{5x-6} = 2 + \sqrt{x-2} & \text{b) } \sqrt{5x-5} = 2 + \sqrt{x-2} \\ \text{c) } \sqrt{5x-4} = 2 + \sqrt{x-2} & \text{d) } \sqrt{5x-14} = 2 + \sqrt{x-2} \end{array}$$

Solution: Consider the equation $\sqrt{5x-a} = 2 + \sqrt{x-2}$. Squaring both sides (this may introduce false roots, so we check at the end) and rearranging: $4x - a - 2 = 4\sqrt{x-2}$. Let $y = \sqrt{x-2} \geq 0$; then $x = y^2 + 2$, and

$$y^2 - y + \frac{6-a}{4} = 0$$

- a) $a = 6$: $y^2 - y = 0$, $y = 0$ or $y = 1$, so $x = 2$ or $x = 3$, and both are indeed solutions.
- b) $a = 5$: $\left(y - \frac{1}{2}\right)^2 = 0$, $y = \frac{1}{2}$, so $x = \frac{9}{4}$, and this is indeed a solution.
- c) $a = 4$: $y^2 - y + \frac{1}{2} = 0$ has no real solution, so the equation has no solution.
- d) $a = 14$: $y^2 - y - 2 = 0$, $y = 2$ ($y = -1$ is negative), so $x = 6$, and this is indeed a solution.

2. The diagonals of the convex quadrilateral $ABCD$ are perpendicular, and the diagonal BD bisects the diagonal AC . Also $\angle ABC = 120^\circ$, $AC = 4\sqrt{3}$, and $BD = 8$. Find the area, the sides, and the angles of the quadrilateral.

Solution: The area is $T = \frac{4\sqrt{3} \cdot 8}{2} = 16\sqrt{3}$ square units.

Let F be the intersection of the diagonals. BD is the perpendicular bisector of AC , so $BA = BC$, and BF bisects the angle ABC . In the right triangle ABF the angle at B is 60° and $AF = 2\sqrt{3}$, so $BF = 2$, $AB = BC = 4$, and $\angle BAF = \angle BCF = 30^\circ$. Since $DF = 8 - 2 = 6$, in the right triangle AFD

$$AD^2 = (2\sqrt{3})^2 + 6^2 = 48, \quad AD = CD = 4\sqrt{3}$$

So $AD = DC = AC$, and the triangle ACD is equilateral. The sides are $AB = BC = 4$ and $CD = DA = 4\sqrt{3}$, and the angles are $\angle A = 90^\circ$, $\angle B = 120^\circ$, $\angle C = 90^\circ$, $\angle D = 60^\circ$.

3. Prove that if $-1 < a < 0$ or $0 < a < 1$, then $\frac{1}{a^2} + \frac{1}{1-a^2} \geq 4$.

Solution: For such a , $a^2(1-a^2) > 0$, and

$$\frac{1}{a^2} + \frac{1}{1-a^2} - 4 = \frac{1-a^2+a^2-4a^2(1-a^2)}{a^2(1-a^2)} = \frac{(2a^2-1)^2}{a^2(1-a^2)} \geq 0$$

Equality holds exactly when $a^2 = \frac{1}{2}$, that is, $a = \pm \frac{1}{\sqrt{2}}$.

4. From which point of the segment joining the centers of the circles $x^2 + y^2 = 9$ and $(x-4)^2 + (y-8)^2 = 1$ can we draw common tangents to the two circles? Find the equations of these tangent lines.

Solution: The centers are $C_1(0,0)$ and $C_2(4,8)$, and the radii are 3 and 1. The tangents through a point of the segment C_1C_2 are the internal common tangents, which pass through the point P_0 dividing C_1C_2 in the ratio of the radii: $\frac{C_1P_0}{P_0C_2} = 3$, so $\overrightarrow{OP_0} = \frac{3}{4}\overrightarrow{OC_2} = (3,6)$.

One common tangent is the line $x = 3$ (its distances from the centers are 3 and 1). The other tangent has a slope m : $y - 6 = m(x - 3)$. Its distance from the origin is 3:

$$\frac{|6 - 3m|}{\sqrt{m^2 + 1}} = 3, \quad m = \frac{3}{4}$$

So the tangents through $P_0(3,6)$ are $x = 3$ and $3x - 4y + 15 = 0$.

5. Find the value of $\frac{y}{x}$, if

a) $\lg^2 y + \lg^2 x - 2 \lg y \cdot \lg x - \lg y + \lg x = 2$; b) $\cos \frac{y+2x}{2x} = \cos \frac{y-2x}{2x}$.

($\lg$ denotes the base 10 logarithm.)

Solution: a) The equation is $(\lg y - \lg x)^2 - (\lg y - \lg x) - 2 = 0$, so $\lg \frac{y}{x} = 2$ or $\lg \frac{y}{x} = -1$. So $\frac{y}{x} = 100$ or $\frac{y}{x} = \frac{1}{10}$.

b) The equation says $\cos\left(\frac{y}{2x} + 1\right) = \cos\left(\frac{y}{2x} - 1\right)$. This holds if $\frac{y}{2x} + 1 = \frac{y}{2x} - 1 + 2k\pi$, that is, $2 = 2k\pi$, which is impossible for an integer k ; or if $\frac{y}{2x} + 1 = -\left(\frac{y}{2x} - 1\right) + 2n\pi$, that is, $\frac{y}{x} = 2n\pi$, where n is an integer.

6. Find the set of ordered pairs (x, y) that satisfy the following system of equations.

$$\sqrt{x^2 - 4x + 4} + \sqrt{y^2 - 6y + 9} = 1, \quad |x - 2| + y = 4$$

Solution: Since $\sqrt{a^2} = |a|$, the system is

$$|x - 2| + |y - 3| = 1, \quad |x - 2| + y = 4$$

Subtracting the second equation from the first: $|y - 3| - y = -3$, that is, $|y - 3| = y - 3$, so $y \geq 3$. Then $|x - 2| = 4 - y \leq 1$, so $1 \leq x \leq 3$, and $y = 4 - |x - 2|$. Conversely, every such pair is a solution. So the solutions are the pairs $(t, t + 2)$ with $1 \leq t \leq 2$ and $(t, 6 - t)$ with $2 \leq t \leq 3$: the points of the graph of $y = 4 - |x - 2|$ with $y \geq 3$.

7. Through the vertices of an equilateral triangle we draw three parallel lines. The distances of the middle line from the other two are 1 and 4. Find the side of the equilateral triangle.

Solution: Let the middle line be f , and the lines at distances 1 and 4 from it be e and g . Let the vertex A of the triangle be on e , B on f , and C on g , and let s be the side of the triangle. Let A_1 be the projection of A on f , A_2 the projection of A on g , and B_1 the projection of B on g . Then $BA_1 = B_1A_2 = B_1C + CA_2$, and since $AA_1 = 1$, $BB_1 = 4$, $AA_2 = 5$:

$$BA_1 = \sqrt{s^2 - 1}, \quad B_1C = \sqrt{s^2 - 16}, \quad CA_2 = \sqrt{s^2 - 25}$$

So

$$\sqrt{s^2 - 1} = \sqrt{s^2 - 16} + \sqrt{s^2 - 25}$$

Squaring: $40 - s^2 = 2\sqrt{s^4 - 41s^2 + 400}$, and squaring again: $3s^4 - 84s^2 = 0$, so $s^2 = 28$. (This satisfies $40 - s^2 \geq 0$.) The side of the triangle is $2\sqrt{7} \approx 5.29$ units.

8. a) Prove the inequality $a^2 + b^2 + c^2 \geq ab + bc + ca$, where a, b, c are real numbers.
 b) Prove that the function $f(x) = (x+a)(x+b) + (x+b)(x+c) + (x+c)(x+a)$ defined on the real numbers has a zero for all real values of a, b , and c .
 c) Prove that the function $g(x) = b^2x^2 + (b^2 + c^2 - a^2)x + c^2$ defined on the real numbers has no zero if a, b , and c are the sides of a triangle.

Solution: a) $(a-b)^2 \geq 0$, so $a^2 + b^2 \geq 2ab$, and similarly $b^2 + c^2 \geq 2bc$, $c^2 + a^2 \geq 2ca$. Adding these and dividing by 2 gives the inequality, with equality exactly when $a = b = c$.

b) $f(x) = 3x^2 + 2(a+b+c)x + ab + bc + ca$, and its discriminant is

$$D = 4(a+b+c)^2 - 12(ab+bc+ca) = 4(a^2 + b^2 + c^2 - ab - bc - ca) \geq 0$$

by part a), so f has a zero.

c) The discriminant of g is

$$\begin{aligned} D_1 &= (b^2 + c^2 - a^2)^2 - 4b^2c^2 = (b^2 + c^2 - a^2 + 2bc)(b^2 + c^2 - a^2 - 2bc) \\ &= ((b+c)^2 - a^2)((b-c)^2 - a^2) = (b+c+a)(b+c-a)(b-c+a)(b-c-a) \end{aligned}$$

By the triangle inequality the first three factors are positive and the fourth is negative, so $D_1 < 0$, and g has no zero. (Since $b^2 > 0$, g is a quadratic function.)