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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Find the equations of the lines through the point $P(4, 1)$ that are perpendicular, and parallel, to the line $3x - 2y = 22$.

Solution: A normal vector of the line $3x - 2y = 22$ is $\mathbf{n}(3, -2)$; this is a direction vector of the perpendicular line, whose equation is $-2x - 3y = -11$, that is, $2x + 3y = 11$. The parallel line is $3x - 2y = 10$.

2. In the first years of the reign of Emperor Trajan, an aqueduct had to be built between the temple of the Vestal Virgins (V) and a new spring (F). It was cheaper to drill through the hill separating the city and the spring than to lay the aqueduct along a route deviating from the straight line. To finish the planned aqueduct faster, the drilling was to start from both sides. Fortunately, Menelaus of Alexandria was staying in the city, and he solved the problem as follows: for the drilling teams to meet, the direction of the line VF must be known. To find it, he chose a base point A that is 5 km from the city and 9 km from the spring, and measured the angle FAV : 97.8° .

a) How long will the aqueduct be?

b) What angle must the ray VF (which gives the place of the drilling) make with the ray VA , and the ray FV with the ray FA ?

Solution: Let $\alpha = \angle FAV$, $\omega = \angle FVA$, and $\varphi = \angle VFA$.

a) By the Law of Cosines in the triangle FAV : $VF^2 = VA^2 + FA^2 - 2 \cdot VA \cdot FA \cdot \cos \alpha = 25 + 81 - 90 \cos 97.8^\circ$, so $VF \approx 10.87$ km.

b) By the Law of Sines: $\frac{\sin \omega}{\sin \alpha} = \frac{FA}{VF}$. Since $\alpha > 90^\circ$, $\omega < 90^\circ$, and $\omega \approx 55.12^\circ$. Since the angles of the triangle add up to 180° , $\varphi \approx 27.08^\circ$.

3. Solve the system of equations $x^{\log_y x} = y^4$, $x + \frac{1}{y^2} = \frac{82}{9}$ over the pairs of positive real numbers different from 1.

Solution: Taking the base y logarithm of both sides of the first equation: $(\log_y x)^2 = 4$, so $\log_y x = -2$ or $\log_y x = 2$, that is, $x = \frac{1}{y^2}$ or $x = y^2$.

In the first case the second equation is $2x = \frac{82}{9}$, so $x_1 = \frac{41}{9}$, and $y_1 = \frac{3}{\sqrt{41}}$ (only the positive value is allowed).

In the second case the second equation is $x + \frac{1}{x} = \frac{82}{9}$, that is, $9x^2 - 82x + 9 = 0$, so $x_2 = \frac{1}{9}$ and $x_3 = 9$, and then $y_2 = \frac{1}{3}$, $y_3 = 3$.

The solutions are $\left(\frac{41}{9}, \frac{3}{\sqrt{41}}\right)$, $\left(\frac{1}{9}, \frac{1}{3}\right)$, and $(9, 3)$.

4. The price of a 36 000 Ft bicycle was reduced by some percent. Since it still did not sell well, after a while it was reduced again by some percent. Both reductions were one-digit integer percents. So the bicycle was sold for 31 806 Ft. By what percent was it reduced the first time, and by what percent the second time?

Solution: Let the first reduction be $x\%$ and the second $y\%$. Then

$$\left(1 - \frac{x}{100}\right) \left(1 - \frac{y}{100}\right) \cdot 36\,000 = 31\,806, \quad (100 - x)(100 - y) = 8835$$

Here $100 - x$ and $100 - y$ are integers between 90 and 100, and $8835 = 3 \cdot 5 \cdot 19 \cdot 31 = 93 \cdot 95$ is the only such factorization. So the bicycle was reduced by 5% and 7% (in either order).

(The original problem asks about a “television” in its last sentence; this is a slip, the item is the bicycle.)

5. The base of the glass pyramid (a right square pyramid) built in the courtyard of the Louvre has side 15. The designers built a cube into the pyramid so that one face of the cube is on the base of the pyramid, and the vertices of the opposite face are on the lateral edges of the pyramid. The side faces and the top of the cube are also made of glass. What is the volume of the glass pyramid, if covering its base square used five times as much material as making the glass walls of the cube?

Solution: Let the edge of the cube be a and the height of the pyramid m . The glass walls of the cube (four side faces and the top) have area $5a^2$, so

$$15^2 = 5 \cdot 5a^2, \quad a = 3$$

By symmetry, the vertices of the top face of the cube are on the lateral edges, above the diagonals of the base. A vertex of the base square is $\frac{15\sqrt{2}}{2}$ from the axis of the pyramid, and a vertex of the top of the cube is $\frac{3\sqrt{2}}{2}$ from the axis, at height 3. By similar triangles (in the plane through the axis and a lateral edge)

$$\frac{\frac{3\sqrt{2}}{2}}{\frac{15\sqrt{2}}{2}} = \frac{m - 3}{m}, \quad m = \frac{15}{4}$$

The volume of the pyramid is $V = \frac{15^2 \cdot \frac{15}{4}}{3} = \frac{1125}{4} = 281.25$ cubic units.

(Note: the KoMaL solution works with a base edge of 14 instead of 15, uses the equation $36a^2 = 196$, and sets up the similarity as $\frac{BC}{AO} = \frac{m - a}{a}$ instead of $\frac{m - a}{m}$; so its answer, $\frac{4802}{27} \approx 177.85$, does not fit the problem as stated. The solution above is ours.)

6. The first term of an arithmetic sequence is -13 . The n th term of the sequence ($1 < n$) is equal to the difference of the sequence, which is a positive integer. Find the sum of the first n terms.

Solution: By the formula for the n th term, $d = -13 + (n - 1)d$, so $(n - 2)d = 13$. Since n and d are positive integers, d is a divisor of 13: $d = 1$ or $d = 13$.

If $d = 1$: $n = 15$, and $S_{15} = \frac{15}{2}(-13 + 1) = -90$.

If $d = 13$: $n = 3$, the terms are $-13, 0, 13$, and $S_3 = 0$.

7. A patient in shock was brought to the emergency room. The heart rhythm analyzer determined that the heart rhythm of the patient changes according to the function $f(x) = \sin^2 2x + 2 \cdot \cos^2 x - \frac{5}{4}$. According to Dr. Green's therapy, the medicine must be given when the heart rhythm takes its smallest and largest values on the interval $[0, \pi]$, and the dose must change according to the smallest and largest values. Unfortunately, the machine broke down then. Fortunately Dr. Carter is also an amateur mathematician, and successfully found the places of the extreme values and the extreme values. What result did he get?

Solution: Since $\sin^2 2x = 4 \sin^2 x \cos^2 x = 4(1 - \cos^2 x) \cos^2 x$,

$$f(x) = -4 \cos^4 x + 6 \cos^2 x - \frac{5}{4} = 1 - \left(2 \cos^2 x - \frac{3}{2}\right)^2$$

The maximum is 1, where $\cos^2 x = \frac{3}{4}$, that is, at $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$ in $[0, \pi]$. The minimum is where $\left|2 \cos^2 x - \frac{3}{2}\right|$ is largest, that is, where $\cos x = 0$: at $x = \frac{\pi}{2}$, and the minimum is $-\frac{5}{4}$.

8. The angle trisectors f_1 and f_2 from the vertex C of the triangle ABC divide the opposite side into the segments c_1, c_2, c_3 (in this order from A). Prove that

$$\frac{f_1}{f_2} = \sqrt{\frac{c_1(c_2 + c_3)}{c_3(c_1 + c_2)}}$$

Solution: Let the angles at A and B be α and β , and let each of the three parts of the angle at C be φ ; f_1 is the trisector next to CA , and f_2 is next to CB . By the Law of Sines in the triangles cut off by the trisectors:

$$\frac{f_1}{c_1} = \frac{\sin \alpha}{\sin \varphi}, \quad \frac{f_2}{c_3} = \frac{\sin \beta}{\sin \varphi}, \quad \text{so} \quad \frac{f_1 c_3}{f_2 c_1} = \frac{\sin \alpha}{\sin \beta}$$

Similarly, in the triangles with angle 2φ at C :

$$\frac{f_2}{c_1 + c_2} = \frac{\sin \alpha}{\sin 2\varphi}, \quad \frac{f_1}{c_2 + c_3} = \frac{\sin \beta}{\sin 2\varphi}, \quad \text{so} \quad \frac{f_2(c_2 + c_3)}{f_1(c_1 + c_2)} = \frac{\sin \alpha}{\sin \beta}$$

The right-hand sides are equal, so $\frac{f_1 c_3}{f_2 c_1} = \frac{f_2(c_2 + c_3)}{f_1(c_1 + c_2)}$, and

$$\frac{f_1}{f_2} = \sqrt{\frac{c_1(c_2 + c_3)}{c_3(c_1 + c_2)}}$$