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translated from: KoMaL, April 2004 supplement, pages 35-36

Pdf file: KoMaL, April 2004 supplement, pages 35-36

In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. For which values of x is it true that

$$\text{a) } 9 - (x - 1)^2 > 0; \quad \text{b) } 2|\sin x| = \frac{3}{\sqrt{3} - 1} - \frac{\sqrt{3}(\sqrt{3} + 1)}{2}; \quad \text{c) both a) and b) hold?}$$

Solution: a) $(x - 1)^2 < 9$, so $-3 < x - 1 < 3$, that is, $-2 < x < 4$.

b) $\frac{3}{\sqrt{3} - 1} = \frac{3(\sqrt{3} + 1)}{2}$, so the right-hand side is $\frac{3\sqrt{3} + 3 - 3 - \sqrt{3}}{2} = \sqrt{3}$. So $|\sin x| = \frac{\sqrt{3}}{2}$, and $x = \pm \frac{\pi}{3} + k\pi$ (k an integer).

c) The values $x = \pm \frac{\pi}{3} + k\pi$ with $-2 < x < 4$ are $x = -\frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}$. ($-\frac{2\pi}{3} \approx -2.09$ and $\frac{4\pi}{3} \approx 4.19$ are outside the interval.)

(Note: In KoMaL, the second term in b) is printed as $\frac{\sqrt{3}(\sqrt{3} - 1)}{2}$; then the right-hand side would be $2\sqrt{3}$, and the equation would have no solution. The published solution uses the value $\sqrt{3}$, so the printed $\sqrt{3} - 1$ is most likely a misprint for $\sqrt{3} + 1$.)

2. The diagonal of a regular pentagon is 8 cm long.

a) What is its side? b) What is its area? c) Into what parts is one of its diagonals divided by the two diagonals that cross it?

Solution: Let the pentagon be $ABCDE$ with center O . The angles of the pentagon are 108° , so in the isosceles triangle BCD the base angles are 36° , and the diagonal is $BD = 2a \cos 36^\circ = 2a \sin 54^\circ$.

$$\text{a) } a = \frac{4}{\sin 54^\circ} \approx 4.94 \text{ cm.}$$

b) The pentagon is made up of five congruent triangles with vertex O , with base a and base angles 54° ; the height of each is $\frac{a}{2} \tan 54^\circ$. So the area is

$$t = 5 \cdot \frac{a}{2} \cdot \frac{a}{2} \tan 54^\circ = \frac{5a^2 \tan 54^\circ}{4} \approx 42.06 \text{ cm}^2$$

c) Let the diagonal BD be crossed by CA at F and by CE at G , with $BF = DG = x$. $ABCD$ is a trapezoid ($AD \parallel BC$), and its diagonals divide each other in the ratio of the parallel sides: the triangles BCF and DAF are similar, so

$$\frac{8 - x}{x} = \frac{AD}{BC} = \frac{8}{a}, \quad x = \frac{8a}{8 + a} \approx 3.06 \text{ cm}$$

So the diagonal is divided into parts of about 3.06, 1.89, and 3.06 cm.

(Note: the KoMaL solution, rounding a to 4.94, gets the area 41.99 cm^2 ; the more precise value is 42.06 cm^2 .)

3. A car travels at a constant speed on a straight road. At two o'clock and at three o'clock the driver sees kilometer signs with two-digit numbers whose digits are the same, in the two different orders. At four o'clock the car passes a sign with a three-digit number. This looks like the number seen two hours earlier, but with a zero between its two digits. Find the speed of the car.

At four o'clock the car turns around and goes back to the sign seen at two o'clock. At what speed must it go so that its average speed for the whole round trip is 20% more than between two and four o'clock?

Solution: a) Let the numbers be $\overline{xy} = 10x + y$, $\overline{yx} = 10y + x$, and $\overline{x0y} = 100x + y$. The car covers equal distances in the two hours:

$$100x + y - (10y + x) = 10y + x - (10x + y), \quad y = 6x$$

so $x = 1$ and $y = 6$: the signs show 16, 61, and 106. The speed is $v = 106 - 61 = 61 - 16 = 45 \text{ km/h}$.

b) From 16 to 106 is 90 km, so the round trip is 180 km. If the speed back is $u \text{ km/h}$, then

$$\frac{180}{2 + \frac{90}{u}} = 45 \cdot \frac{6}{5} = 54, \quad u = 67.5 \text{ km/h}$$

4. A bank offers two kinds of investments.

The first: the part of the deposit below 1 million Ft earns 5.5% interest, the part between 1 million Ft and 3 million Ft earns 6%, and the part above 3 million Ft earns 7%, credited yearly.

The second: 1.6% interest every three months, credited quarterly.

Compare the two investments as a function of the deposit.

Solution: In the second investment the yearly growth factor is $1.016^4 \approx 1.0656$, that is, about 6.56% per year. For deposits up to 3 million Ft the first investment pays at most 6% on every part, so the second one is better. So we compute, in millions of Ft, how 3 million plus an additional x million grows in two years (as in the KoMaL solution).

First investment: after one year 3 million grows to $1 \cdot 1.055 + 2 \cdot 1.06 = 3.175$, and after two years (the interest is again $0.055 + 0.12 + 0.175 \cdot 0.07$) to 3.36225. The additional x earns 7% both years. So

$$T(x) = 1.07^2 x + 3.36225 = 1.1449x + 3.36225$$

Second investment: $t(x) = 1.016^8 (x + 3) \approx 1.135402x + 3.406206$.

$T(x) = t(x)$ at $x \approx 4.628$. So over two years the first investment is better for deposits greater than about 7.63 million Ft, and the second is better for smaller deposits.

(Note: the KoMaL solution gets $x \approx 4.596$ and states the break-even point as 4 596 336 Ft, forgetting to add the 3 million; the break-even deposit is about $3 + 4.628 = 7.628$ million Ft. Over a one-year period the break-even deposit is about 7.87 million Ft, since $0.175 + 0.07(D - 3) = (1.016^4 - 1)D$ gives $D \approx 7.87$.)

5. Prove the following statements.

a) Every prime number greater than 3 has a neighbor divisible by 6.

b) $n^4 + 4$ is a composite number if n is an integer greater than 1.

Solution: a) A prime $p > 3$ is odd, so both of its neighbors $p - 1$ and $p + 1$ are even. Of the three consecutive integers $p - 1, p, p + 1$, one is divisible by 3, and it is not p . So one of $p - 1$ and $p + 1$ is divisible by both 2 and 3, that is, by 6.

b) $n^4 + 4 = (n^2 + 2)^2 - 4n^2 = (n^2 + 2 - 2n)(n^2 + 2 + 2n)$, and the smaller factor is $n^2 - 2n + 2 = (n - 1)^2 + 1 > 1$ for $n > 1$. The other factor is even larger, so the product is composite.

6. Six boys and four girls study in an English language group. The teacher calls on three students. What is the probability that both boys and girls are among the students called on?

Solution: Both boys and girls are called on if two boys and one girl, or one boy and two girls are called on. The number of such choices is

$$\binom{6}{2} \cdot 4 + \binom{4}{2} \cdot 6 = 60 + 36 = 96$$

The total number of choices is $\binom{10}{3} = 120$, so the probability is $\frac{96}{120} = 0.8$.

7. What is the range of the function $f(x) = \frac{1}{\sin^4 x + \cos^4 x}$ defined on the real numbers? Find the places of the maximum and the minimum.

Solution: $\sin^4 x + \cos^4 x = (\sin^2 x + \cos^2 x)^2 - 2 \sin^2 x \cos^2 x = 1 - \frac{\sin^2 2x}{2}$, so

$$f(x) = \frac{2}{2 - \sin^2 2x}$$

Since $1 \leq 2 - \sin^2 2x \leq 2$, we get $1 \leq f(x) \leq 2$, and the range is $[1, 2]$. The maximum 2 is taken where $\sin^2 2x = 1$, that is, at $x = \frac{\pi}{4} + \frac{k\pi}{2}$, and the minimum 1 where $\sin 2x = 0$, that is, at $x = \frac{k\pi}{2}$ (k an integer).

8. Every member of a group speaks at least one of French and German. 70% of them speak German and 60% speak French.

- What is the probability that a randomly chosen person cannot answer in German?
- How many people speak only one language, if 12 people do not speak French?
- What is the answer to a), under the condition of b), if the group grows by 6 members who speak both languages?

Solution: Since $70\% + 60\% = 130\%$, 30% of the group speaks both languages, 40% speaks only German, and 30% only French.

- The people who cannot answer in German are those who speak only French: $p = 0.3$.
- The 12 people who do not speak French are those who speak only German, which is 40%, so the group has 30 members; 9 speak only French, and $12 + 9 = 21$ people speak only one language.
- Now 12 speak only German, 9 only French, and $9 + 6 = 15$ both languages; the group has 36 members. The probability is $\frac{9}{36} = 0.25$.

9. In a department store, two rectangular sales areas of equal area ("A" and "B") are to be created. The available space and operating conditions require that

- the width of area "A" is 8 m less than its length;
- the width of area "B" is 1 m less than the width of area "A";
- the length of area "B" is as small as possible.

Give the dimensions of the two areas.

Solution: Let the width of "A" be x (so its length is $x + 8$), the width of "B" $x - 1$, and its length y ($x > 1$). Since the areas are equal, $(x - 1)y = x(x + 8)$, so

$$y = \frac{x^2 + 8x}{x - 1} = (x - 1) + \frac{9}{x - 1} + 10$$

By the inequality between the arithmetic and geometric means, $(x-1) + \frac{9}{x-1} \geq 2\sqrt{9} = 6$, with equality for $x-1 = 3$. So y is smallest, 16 m, for $x = 4$. Area “A” is 4 m by 12 m, and area “B” is 3 m by 16 m.