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In memory of György Ágoston (Miskolc, Szeged), Károly Kellner, Tamás Holczer, and my other friends who were killed in the Holocaust

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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance. The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve the following equations over the real numbers.

$$\text{a) } \sin\left(3x - \frac{\pi}{3}\right) = \sin\left(2x + \frac{\pi}{6}\right) \quad \text{b) } \sin\left(3x - \frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3} - 2x\right)$$

$$\text{c) } \cos\left(3x - \frac{5\pi}{6}\right) = \sin\left(2x + \frac{\pi}{6}\right) \quad \text{d) } \cos\left(\frac{5\pi}{6} - 3x\right) = \cos\left(2x - \frac{\pi}{3}\right)$$

Solution: $\sin \alpha = \sin \beta$ exactly when $\alpha = \beta + 2k\pi$ or $\alpha = \pi - \beta + 2k\pi$. Using $\sin \alpha = \cos\left(\frac{\pi}{2} - \alpha\right)$, $\cos \alpha = \cos(-\alpha)$, and $\sin \alpha = \sin(\pi - \alpha)$, all four equations are equivalent to equation a). (For example $\cos\left(\frac{\pi}{3} - 2x\right) = \sin\left(2x + \frac{\pi}{6}\right)$, and $\cos\left(3x - \frac{5\pi}{6}\right) = \sin\left(\frac{4\pi}{3} - 3x\right) = \sin\left(3x - \frac{\pi}{3}\right)$.) So

$$3x - \frac{\pi}{3} = 2x + \frac{\pi}{6} + 2k\pi \quad \text{or} \quad 3x - \frac{\pi}{3} = \frac{5\pi}{6} - 2x + 2k\pi$$

and the solutions of all four equations are

$$x = \frac{\pi}{2} + 2k\pi \quad \text{or} \quad x = \frac{7\pi}{30} + \frac{2k\pi}{5}, \quad k \text{ is an integer}$$

(Note: the KoMaL solution gives $\frac{7\pi}{30} + 2k\pi$ in the second case; this is a slip, since $5x = \frac{7\pi}{6} + 2k\pi$ gives $x = \frac{7\pi}{30} + \frac{2k\pi}{5}$.)

2. Solve the equation $\sqrt{x} + \sqrt{x - 4a + 4} = 2$ over the real numbers, where a is a real parameter.

Solution: Suppose there is a solution. Then $(\sqrt{x - 4a + 4})^2 = (2 - \sqrt{x})^2$, so $x - 4a + 4 = 4 - 4\sqrt{x} + x$, $\sqrt{x} = a$, and $x = a^2$. This is a solution if it satisfies the equation:

$$\sqrt{a^2} + \sqrt{a^2 - 4a + 4} = 2, \quad |a| + |a - 2| = 2$$

If $a < 0$, this is $2 - 2a = 2$, $a = 0$, a contradiction. If $0 \leq a \leq 2$, it is $a + 2 - a = 2$, true for every such a . If $a > 2$, it is $2a - 2 = 2$, $a = 2$, a contradiction. So the equation has a solution exactly when $0 \leq a \leq 2$, and then the solution is $x = a^2$.

3. The altitude to the hypotenuse AB of the right triangle ABC is CD . The radii of the circles inscribed in the triangles ACD and BCD are $\rho_1 = 3$ and $\rho_2 = 4$. Find the radius of the circle inscribed in the triangle ABC .

Solution: The right triangles ACD , CBD , and ABC are similar, and their hypotenuses are AC , BC , AB . So the ratios of corresponding lengths are equal: there is a k with $AC = 3k$, $BC = 4k$, $AB = k\rho$, where ρ is the radius of the circle inscribed in ABC . By the Pythagorean Theorem

$$k^2 \rho^2 = 9k^2 + 16k^2, \quad \rho^2 = 25, \quad \rho = 5$$

(Such a triangle exists, for example $AC = 15$, $BC = 20$, $AB = 25$.)

4. The equation of the median through the vertex B of the triangle ABC is $3x + 5y = 10$, and the equation of the median through C is $3x - 2y = 3$. One vertex of the triangle is $(-3, 1)$. Find the coordinates of the other two vertices.

Solution: The point $(-3, 1)$ is on neither median, so it is $A(-3, 1)$. The medians meet at the centroid $S\left(\frac{5}{3}, 1\right)$. Let

A_1 be the midpoint of BC . Since $\overrightarrow{AS} = 2\overrightarrow{SA_1}$ and $\overrightarrow{AS} = \left(\frac{14}{3}, 0\right)$, we have $\overrightarrow{SA_1} = \left(\frac{7}{3}, 0\right)$, and $A_1(4, 1)$.

The points of the median s_B are $(5t, 2 - 3t)$, and the points of s_C are $(1 + 2v, 3v)$. A_1 is the midpoint of BC , so $5t + 1 + 2v = 8$ and $2 - 3t + 3v = 2$, from which $t = v = 1$. So $B(5, -1)$ and $C(3, 3)$.

5. Solve the following equations over the real numbers.

a) $\lg^2(5 - x) + 2\lg^2(x + 1) = 3\lg(5 - x) \cdot \lg(x + 1)$

b) $3^{2x+1} + 3^{2-2\sqrt{x}} = 4 \cdot 3^{1+x-\sqrt{x}}$

c) $2\sin^4 x - 2\sin^2 x \cos^2 x + 4\cos^4 x = 1$

Solution: a) We need $-1 < x < 5$. Let $a = \lg(5 - x)$ and $b = \lg(x + 1)$. Then $a^2 - 3ab + 2b^2 = (a - b)(a - 2b) = 0$, so $a = b$ or $a = 2b$. From $5 - x = x + 1$, $x_1 = 2$; from $5 - x = (x + 1)^2$, $x = 1$ or $x = -4$, and only $x_2 = 1$ is in the domain. The solutions are $x_1 = 2$ and $x_2 = 1$.

b) We need $x \geq 0$. Let $a = 3^x$ and $b = 3^{-\sqrt{x}}$. Dividing by 3 and rearranging, $a^2 - 4ab + 3b^2 = (a - b)(a - 3b) = 0$, so $3^x = 3^{-\sqrt{x}}$ or $3^x = 3^{1-\sqrt{x}}$, that is, $x = -\sqrt{x}$ or $x = 1 - \sqrt{x}$. The first gives $x_1 = 0$; the second gives $\sqrt{x} = \frac{-1 + \sqrt{5}}{2}$ (since $\sqrt{x} \geq 0$), so $x_2 = \frac{3 - \sqrt{5}}{2}$.

c) Using $1 = (\sin^2 x + \cos^2 x)^2 = \sin^4 x + 2\sin^2 x \cos^2 x + \cos^4 x$, the equation becomes $\sin^4 x - 4\sin^2 x \cos^2 x + 3\cos^4 x = 0$. If $\cos x = 0$, then $\sin^2 x = 1$, which is not a solution, so we can divide by $\cos^4 x$: $\tan^4 x - 4\tan^2 x + 3 = 0$, so $\tan^2 x = 1$ or $\tan^2 x = 3$. The solutions are

$$x = \frac{\pi}{4} + \frac{k\pi}{2}, \quad x = \frac{\pi}{3} + n\pi, \quad x = -\frac{\pi}{3} + m\pi, \quad k, n, m \text{ integers}$$

6. For which integers n is the value of the expression

$$\frac{18nx + 36x + 12n + 6n^2}{3n^2x + 15nx + 18x + 6n + 5n^2 + n^3}$$

an integer, if x can be any integer?

Solution: Factoring the numerator and the denominator:

$$\frac{6(n+2)(3x+n)}{(n+2)(n+3)(3x+n)} = \frac{6}{n+3}$$

where $n \neq -2$, $n \neq -3$, and n cannot be a multiple of 3 (otherwise $n = -3x$ for some integer x , and the denominator is 0). So $n+3$ must be a divisor of 6, and $n \neq -2$, and n is not a multiple of 3. If $n+3 = -1, 2$, or -2 , then $n = -4, -1$, or -5 , and these are the solutions. (If $n+3$ is 1, 3, -3 , 6, or -6 , then n is -2 or a multiple of 3.) For $n = -1$ the value is 3, for $n = -4$ it is -6 , and for $n = -5$ it is -3 .

7. For which values of the real parameter p is the sum of the squares of the real roots of the equation $x^2 + (2p - 1)x + \frac{3}{2}p^2 - 5p + \frac{1}{4} = 0$ smallest, and for which is it largest? What are the smallest and largest values?

Solution: The roots are real if the discriminant is nonnegative:

$$D = (2p - 1)^2 - 4 \left(\frac{3}{2}p^2 - 5p + \frac{1}{4} \right) = -2p^2 + 16p = 2(16 - (p - 4)^2) \geq 0$$

that is, $0 \leq p \leq 8$. By Viète's formulas

$$f(p) = x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1x_2 = (2p - 1)^2 - 2 \left(\frac{3}{2}p^2 - 5p + \frac{1}{4} \right) = (p + 3)^2 - \frac{17}{2}$$

For $0 \leq p \leq 8$, $9 \leq (p + 3)^2 \leq 121$, so $\frac{1}{2} \leq f(p) \leq 112.5$. The smallest value is $\frac{1}{2}$, at $p = 0$, and the largest value is 112.5, at $p = 8$.

8. The points K on the side AB and L on the side BC of the triangle ABC satisfy $3AK = KB$ and $BL = LC$. The segments AL and CK meet at Q .

- a) What fraction of the area of the triangle ABC is the area of the triangle AQB , and of the triangle BQC ?
b) In what ratio does Q divide the segments AL and CK ?

Solution: Join Q and B . Let t_1 be the area of the triangle AQK , t_2 the area of QBL , t the area of QCA , and T the area of ABC . By the conditions, the area of KQB is $3t_1$, the area of QLC is t_2 , and

$$\frac{T}{2} = 4t_1 + t_2, \quad \frac{3}{4}T = 3t_1 + 2t_2, \quad \frac{T}{4} = t + t_1$$

(the triangles ABL , KBC , and AKC). From the first two equations, $8t_1 + 2t_2 = 4t_1 + \frac{8}{3}t_2$, so $t_2 = 6t_1$, $T = 20t_1$, and then $t = 4t_1$.

- a) $\frac{t_{AQB}}{T} = \frac{4t_1}{20t_1} = \frac{1}{5}$ and $\frac{t_{BQC}}{T} = \frac{2t_2}{20t_1} = \frac{3}{5}$.
b) $\frac{AQ}{QL} = \frac{t}{t_2} = \frac{4t_1}{6t_1} = \frac{2}{3}$ and $\frac{CQ}{QK} = \frac{t}{t_1} = 4$.