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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. Show that there is no pair of real numbers (x, y) for which

$$(x^2 + 5)^2 = 25 - |y - 3|, \quad 21x + 63y = 188$$

(11 points)

Solution: In the first equation $(x^2 + 5)^2 \geq 25$ and $25 - |y - 3| \leq 25$, so it can hold only for $x = 0, y = 3$. But $21 \cdot 0 + 63 \cdot 3 = 189 \neq 188$, so the system has no solution.

2. Consider the function $f(x) = ax^2 + (2a + b)x - (b^2 - b - a)$ defined on the real numbers.

- a) Show that if a and b are integers, then $f(-2) + f(-1) + f(0) + f(1) + f(2)$ is divisible by 5.
b) Prove that if a is positive, then $f(x)$ has a zero.

(12 points)

Solution: a) This is true for every polynomial $f(x) = Ax^2 + Bx + C$ with integer coefficients: $f(-2) + f(2) = 8A + 2C$, $f(-1) + f(1) = 2A + 2C$, and $f(0) = C$, so the sum is $10A + 5C$, which is divisible by 5.

b) The discriminant is $(2a + b)^2 + 4a(b^2 - b - a) = b^2 + 4ab^2 = b^2(1 + 4a)$, which is nonnegative if $a > 0$. So f (a quadratic function, since $a \neq 0$) has a zero.

3. How many positive terms less than 2 does the sequence $a_n = -5 + \log_2(n + 4)$ have?

(14 points)

Solution: We need the positive integers n with $0 < -5 + \log_2(n + 4) < 2$, that is, $5 < \log_2(n + 4) < 7$, or $\log_2 32 < \log_2(n + 4) < \log_2 128$. The base 2 logarithm is increasing, so $32 < n + 4 < 128$, that is, $28 < n < 124$. So $n = 29, 30, \dots, 123$: the 95 terms $a_{29}, \dots, a_{123}$ satisfy the conditions.

4. On March 25, 2005, Ágnes deposits 600 000 Ft in a bank where the yearly interest is 8%, and the interest is computed at the end of the calendar year. How much will she have on March 25, 2006?

(14 points)

Solution: (Following the KoMaL solution, we assume that the money grows exponentially during the year.) March 25 is the 84th day of the year, so in 2005 the money earns interest for 281 days, and at the end of the year it grows to $1.08^{\frac{281}{365}}$ times its value; in the remaining 84 days it grows to $1.08^{\frac{84}{365}}$ times that. So the year-end break does not matter, and on March 25, 2006 Ágnes will have

$$600\,000 \cdot 1.08 = 648\,000 \text{ Ft}$$

(If the bank computes interest proportionally to the time (simple interest within the year), then at the end of 2005 she has $600\,000 \left(1 + 0.08 \cdot \frac{281}{365}\right) \approx 636\,953$ Ft, and by March 25, 2006 this earns about $636\,953 \cdot 0.08 \cdot \frac{84}{365} \approx 11\,727$ Ft more interest, which is only credited at the end of 2006.)

Part II

5. Three lines are given by their equations:

$$e: x - \sqrt{3}y = 0, \quad f: \sqrt{3}x - y = 0, \quad g: \sqrt{3}x + y - 10 = 0$$

- Show that the three lines determine a right triangle.
- What is the radius of the circle circumscribed around the triangle?
- Find the coordinates of the center of the inscribed circle.

(16 points)

Solution: a) e and f pass through the origin, g does not, and no two of the lines are parallel, so the lines determine a triangle. The slope of e is $\frac{1}{\sqrt{3}}$ and the slope of g is $-\sqrt{3}$; their product is -1 , so e and g are perpendicular, and the triangle is a right triangle.

b) The vertices are $B(0, 0)$ (e and f), $A\left(\frac{5}{\sqrt{3}}, 5\right)$ (f and g), and $C\left(\frac{5\sqrt{3}}{2}, \frac{5}{2}\right)$ (e and g , the right angle). By Thales' theorem the radius of the circumscribed circle is half the hypotenuse $AB = \frac{10}{\sqrt{3}}$: $r = \frac{5}{\sqrt{3}} \approx 2.89$.

c) Let D be the projection of A onto the x -axis. The direction angles of the lines e and f are 30° and 60° , so $\angle ABC = 30^\circ$, and the bisector from B has direction angle 45° : it is the line $y = x$. In the right triangle BDA , $\angle BAD = 30^\circ$, and in the triangle ABC , $\angle BAC = 60^\circ$, so the bisector from A is the line AD , that is, $x = \frac{5}{\sqrt{3}}$. The

center of the inscribed circle is on both bisectors: $\left(\frac{5}{\sqrt{3}}, \frac{5}{\sqrt{3}}\right) \approx (2.89, 2.89)$.

6. Consider the function $f(x) = 2x^6 - 3x^4 + x^2$ defined on the real numbers.

- Find the exact value of $f\left(\tan \frac{\pi}{3}\right)$.
- Show that $f(\sin \alpha) + f(\cos \alpha)$ does not depend on α .

(16 points)

Solution: a) $f(\sqrt{3}) = 2 \cdot 27 - 3 \cdot 9 + 3 = 30$.

b) Let $s = \sin^2 \alpha$ and $c = \cos^2 \alpha$; then $s + c = 1$, $s^2 + c^2 = 1 - 2sc$, and $s^3 + c^3 = (s + c)(s^2 - sc + c^2) = 1 - 3sc$. So

$$f(\sin \alpha) + f(\cos \alpha) = 2(s^3 + c^3) - 3(s^2 + c^2) + (s + c) = 2(1 - 3sc) - 3(1 - 2sc) + 1 = 0$$

which does not depend on α .

7. In a high school, 75% of the students do their math homework. A journalist wants to talk with five randomly chosen students of this school about their study habits.
- What is the probability that he chooses only students who have done their homework?
 - What is the probability that at least three of the five chosen students have done their homework?
 - In class 12.c of the school, with 20 students (where the situation is a little better than the school average), 16 wrote their homework. The class has 3 girls, who always do their homework. If we choose 4 boys and 1 girl from this class, what is the probability that exactly two of those chosen have not done their homework?

(16 points)

Solution: a) Each student has done the homework with probability 0.75; assuming the students are independent, the probability is $0.75^5 \approx 0.2373$.

b) Exactly five: $0.75^5 \approx 0.2373$; exactly four: $\binom{5}{4} 0.75^4 \cdot 0.25 \approx 0.3955$; exactly three: $\binom{5}{3} 0.75^3 \cdot 0.25^2 \approx 0.2637$. The probability is the sum, about 0.8965.

c) There are 17 boys and 3 girls, and 4 boys did not do the homework. 4 boys and 1 girl can be chosen in $\binom{17}{4} \cdot \binom{3}{1} = 7140$ ways. The girl certainly did the homework, so the two students without homework are boys: we choose 2 of the 13 boys with homework and 2 of the 4 boys without, in $3 \cdot \binom{13}{2} \cdot \binom{4}{2} = 1404$ ways. The probability is $\frac{1404}{7140} \approx 0.1966$.

8. Walking from the bank of a 27 m wide river perpendicular to the bank, we measured the depth of the water every 3 meters. We got the following data, in centimeters: 60, 120, 150, 240, 210, 180, 90, 30.
- What is the area of the cross section, in m^2 , if we assume that the river bed is a straight line between two measuring points?
 - Find the mean and the median of the measured data.
 - What is the average depth of the river along the line of the measurement?
 - How much water flows through the cross section of the river in 1 hour, if the speed of the river is 85 m/min?

(16 points)

Solution: a) The cross section consists of two triangles (at the banks) and seven trapezoids, each 3 m wide:

$$T = 3 \left(\frac{0.6}{2} + \frac{0.6 + 1.2}{2} + \frac{1.2 + 1.5}{2} + \frac{1.5 + 2.4}{2} + \frac{2.4 + 2.1}{2} + \frac{2.1 + 1.8}{2} + \frac{1.8 + 0.9}{2} + \frac{0.9 + 0.3}{2} + \frac{0.3}{2} \right) = 32.4 \text{ m}^2$$

- b) The mean of the data is 1.35 m. In increasing order the two middle values are 1.2 and 1.5, so the median is also 1.35 m.
- c) Dividing the area of the cross section by the width of the river, the average depth is $\frac{32.4}{27} = 1.2$ m.
- d) In 1 hour the water flows $85 \cdot 60 = 5100$ m, so $5100 \cdot 32.4 = 165\,240 \text{ m}^3$ of water flows through the cross section per hour.

9. Among the convex quadrilaterals with the same perimeter, for which is the product of the sums of the opposite sides largest? Find this largest value.

(16 points)

Solution: Let the sides be a, b, c, d (in this order), and the perimeter $k = a + b + c + d$. By the inequality between the arithmetic and geometric means of the positive numbers $a + c$ and $b + d$:

$$\sqrt{(a + c)(b + d)} \leq \frac{(a + c) + (b + d)}{2} = \frac{k}{2}, \quad (a + c)(b + d) \leq \frac{k^2}{4}$$

with equality exactly when $a + c = b + d$. So the product is largest, $\frac{k^2}{4}$, for the quadrilaterals in which the sums of the opposite sides are equal, that is, for the tangential quadrilaterals (those that have an inscribed circle).