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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Consider the right circular cylinders in which the sum of the radius of the base and the height is 10 cm. Which of these cylinders has the largest lateral surface area? What is the volume of the cylinder in this case?

Solution: For real numbers a and b , $0 \leq (a - b)^2$, so $4ab \leq (a + b)^2$, that is, $ab \leq \left(\frac{a + b}{2}\right)^2$, with equality exactly when $a = b$. Let the radius of the base be r and the height m ; the lateral surface area is $P = 2\pi rm$, and $r + m = 10$. So

$$P = 2\pi rm \leq 2\pi \left(\frac{r + m}{2}\right)^2 = 2\pi \cdot 5^2 = 50\pi$$

The lateral surface area is largest, $50\pi \text{ cm}^2$, for $r = m = 5 \text{ cm}$, and then the volume is $V = 5^2 \cdot \pi \cdot 5 = 125\pi \approx 392.7 \text{ cm}^3$.

2. Solve the following system of linear equations, where a is a real parameter.

$$(a - 1)x + 3y = 3a, \quad x + (a + 1)y = 2$$

Solution: From the second equation $x = 2 - (a + 1)y$, and substituting into the first: $(4 - a^2)y = a + 2$.

a) If $|a| \neq 2$: $y = \frac{1}{2 - a}$ and $x = \frac{3(a - 1)}{a - 2}$.

b) If $a = -2$: the equations are $-3x + 3y = -6$ and $x - y = 2$, so the solutions are $x = t + 2$, $y = t$, where t is any real number.

c) If $a = 2$: the equations are $x + 3y = 6$ and $x + 3y = 2$, which contradict each other, so there is no solution.

3. The area of an isosceles trapezoid is $144\sqrt{3} \text{ cm}^2$, and its legs are equal to the radius of the circle circumscribed around the trapezoid. Find the height, the diagonal, and the midline of the trapezoid. Can the parallel sides of the trapezoid be determined?

Solution: Let the trapezoid be $ABCD$ with legs $AD = BC = r$, where r is the radius of the circumscribed circle with center O , and $AB > DC$. All three sides of the triangle OBC are r , so the central angle of the shorter arc BC is 60° , and the inscribed angle $CAB = 30^\circ$. Let C_1 be the projection of C on AB . Then $CC_1 = m$ is the height, and AC_1 is equal to the midline of the trapezoid (in an isosceles trapezoid $AC_1 = \frac{AB + DC}{2}$). In the right triangle AC_1C with $\angle CAC_1 = 30^\circ$: $AC = 2m$ and $AC_1 = m\sqrt{3}$. So the area is

$$m\sqrt{3} \cdot m = 144\sqrt{3}, \quad m = 12 \text{ cm}$$

The midline is $AC_1 = 12\sqrt{3} \approx 20.78 \text{ cm}$ and the diagonal is $AC = 24 \text{ cm}$. There are infinitely many such trapezoids, so the parallel sides cannot be determined.

4. Solve the following equations over the real numbers.

a) $\sqrt{4 \cdot 2^{2x} - 12 \cdot 2^x + 9} = 3 - 2^{x+1}$ b) $\sqrt{2 \cos 2x - 4 \cos x + 3} = 1 - 2 \cos x$

Solution: a) $4 \cdot 2^{2x} - 12 \cdot 2^x + 9 = (2^{x+1} - 3)^2$, so the equation is $|2^{x+1} - 3| = 3 - 2^{x+1}$, which holds exactly when $2^{x+1} \leq 3$, that is,

$$x \leq \log_2 \frac{3}{2} \approx 0.585$$

b) $2 \cos 2x - 4 \cos x + 3 = 2(2 \cos^2 x - 1) - 4 \cos x + 3 = (2 \cos x - 1)^2$, so the equation is $|2 \cos x - 1| = 1 - 2 \cos x$, which holds exactly when $\cos x \leq \frac{1}{2}$, that is,

$$\frac{\pi}{3} + 2k\pi \leq x \leq \frac{5\pi}{3} + 2k\pi, \quad k \text{ is an integer}$$

5. S_n , S_k , and S_{n+k} are the sums of the first n , k , and $n + k$ terms of an arithmetic sequence. Prove that

$$(n + k)(S_n - S_k) = (n - k) \cdot S_{n+k}$$

Solution: Let the first term be a_1 and the difference d . Then

$$\begin{aligned} S_n - S_k &= \frac{n}{2}(2a_1 + (n-1)d) - \frac{k}{2}(2a_1 + (k-1)d) = (n-k)a_1 + \frac{n^2 - n - k^2 + k}{2}d \\ &= \frac{n-k}{2}(2a_1 + (n+k-1)d) \end{aligned}$$

so

$$(n + k)(S_n - S_k) = (n - k) \cdot \frac{n + k}{2}(2a_1 + (n + k - 1)d) = (n - k) \cdot S_{n+k}$$

6. Consider the function $x \mapsto f(x)$ defined on the interval $[-4, 4]$, where

$$f(x) = \frac{|x-3| + |x+1|}{|x+3| + |x-1|}$$

Find the maximum and the minimum of the function, and the values of x where they are taken.

Solution: The denominator is positive for every x , so f is defined on $[-4, 4]$.

a) If $-4 \leq x \leq -3$: $f(x) = \frac{3-x-x-1}{-3-x-x+1} = 1 - \frac{2}{x+1}$, increasing, with $f(-4) = \frac{5}{3} \leq f(x) \leq 2 = f(-3)$.

b) If $-3 < x < -1$: $f(x) = \frac{2-2x}{4} = \frac{1-x}{2}$, decreasing, with $2 > f(x) > 1$.

c) If $-1 \leq x \leq 1$: $f(x) = 1$.

d) If $1 < x < 3$: $f(x) = \frac{4}{2x+2} = \frac{2}{x+1}$, decreasing, with $1 > f(x) > \frac{1}{2}$.

e) If $3 \leq x \leq 4$: $f(x) = \frac{2x-2}{2x+2} = 1 - \frac{2}{x+1}$, increasing, with $f(3) = \frac{1}{2} \leq f(x) \leq \frac{3}{5} = f(4)$.

The largest value of f is 2, at $x = -3$, and the smallest value is $\frac{1}{2}$, at $x = 3$.

7. Prove that if α and β are two angles of a triangle and

$$\sin(\alpha - \beta) = \sin^2 \alpha - \sin^2 \beta$$

then the triangle is a right triangle or an isosceles triangle.

Solution: Using the sum-to-product formulas:

$$\begin{aligned} \sin^2 \alpha - \sin^2 \beta &= (\sin \alpha + \sin \beta)(\sin \alpha - \sin \beta) = \left(2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}\right) \left(2 \cos \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}\right) \\ &= \sin(\alpha + \beta) \cdot \sin(\alpha - \beta) \end{aligned}$$

So the condition is $\sin(\alpha - \beta)(1 - \sin(\alpha + \beta)) = 0$. If $\sin(\alpha - \beta) = 0$, then (since $-180^\circ < \alpha - \beta < 180^\circ$) $\alpha = \beta$, and the triangle is isosceles. If $\sin(\alpha + \beta) = 1$, then (since $0^\circ < \alpha + \beta < 180^\circ$) $\alpha + \beta = 90^\circ$, and the triangle is a right triangle.

8. Notice that

$$44 - 8 = 6^2, \quad 4444 - 88 = 66^2, \quad 444444 - 888 = 666^2$$

Prove that

$$\underbrace{44 \dots 4}_{2n \text{ digits}} - \underbrace{88 \dots 8}_{n \text{ digits}} = \underbrace{66 \dots 6}_{n \text{ digits}}^2$$

Solution: Let $u = \underbrace{11 \dots 1}_{n \text{ digits}}$; then $10^n - 1 = 9u$. The $2n$ -digit number is $4u \cdot 10^n + 4u$, and the n -digit number of 8s is

$8u$. So

$$4u \cdot 10^n + 4u - 8u = 4u(10^n - 1) = 4u \cdot 9u = 36u^2 = (6u)^2$$

and $6u = \underbrace{66 \dots 6}_{n \text{ digits}}$.