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*In memory of my parents and my brother*

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In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance. The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve the following equations over the real numbers.

a)  $(2y - 1)(y - 1) = 0$

b)  $2^{2x+1} - 3 \cdot 2^x + 1 = 0$

c)  $2 \sin^2 x - 3 \sin x + 1 = 0$

d)  $2 \cdot \log_2^2 x - \log_2 x^3 + 1 = 0$

Solution: a)  $y_1 = \frac{1}{2}, y_2 = 1$ .

b) With  $y = 2^x$  this is  $2y^2 - 3y + 1 = 0$ , so  $2^x = \frac{1}{2}$  or  $2^x = 1$ :  $x_1 = -1, x_2 = 0$ .

c) With  $y = \sin x$ :  $\sin x = \frac{1}{2}$  or  $\sin x = 1$ , so  $x = \frac{\pi}{6} + 2n\pi, x = \frac{5\pi}{6} + 2k\pi$ , or  $x = \frac{\pi}{2} + 2m\pi$  ( $n, k, m$  integers).

d) With  $y = \log_2 x$  (and  $\log_2 x^3 = 3 \log_2 x$ ):  $\log_2 x = \frac{1}{2}$  or  $\log_2 x = 1$ , so  $x_1 = \sqrt{2}, x_2 = 2$ .

2. Solve the following inequalities over the real numbers.

a)  $2y^2 - 3y + 1 > 0$

b)  $2^{2x+1} - 3 \cdot 2^x + 1 > 0$

c)  $2 \sin^2 x - 3 \sin x + 1 < 0$

d)  $2 \cdot \log_2^2 x - \log_2 x^3 + 1 > 0$

Solution: a)  $y < \frac{1}{2}$  or  $y > 1$ .

b)  $2^x < \frac{1}{2}$  or  $2^x > 1$ , so  $x < -1$  or  $x > 0$ .

c)  $\frac{1}{2} < \sin x < 1$ , so  $\frac{\pi}{6} + 2k\pi < x < \frac{\pi}{2} + 2k\pi$  or  $\frac{\pi}{2} + 2k\pi < x < \frac{5\pi}{6} + 2k\pi$  ( $k$  an integer).

d)  $\log_2 x < \frac{1}{2}$  or  $\log_2 x > 1$ , so  $0 < x < \sqrt{2}$  or  $x > 2$ .

3. The endpoints of one leg of an isosceles right triangle are  $A(-2, 3)$  and  $B(1, 2)$ . Find the coordinates of the third vertex of the triangle.

Solution: Four isosceles right triangles satisfy the conditions: construct the squares  $ABCD$  and  $ABC_1D_1$  on the two sides of the segment  $AB$ ; the third vertex can be any of  $C, D, C_1, D_1$ .  $\overrightarrow{AB} = (3, -1)$ , and its rotations by  $90^\circ$  are  $(1, 3)$  and  $(-1, -3)$ , so

$$\begin{aligned}\overrightarrow{OC} &= \overrightarrow{OB} + (1, 3) = (2, 5), & \overrightarrow{OC_1} &= \overrightarrow{OB} + (-1, -3) = (0, -1) \\ \overrightarrow{OD} &= \overrightarrow{OA} + (1, 3) = (-1, 6), & \overrightarrow{OD_1} &= \overrightarrow{OA} + (-1, -3) = (-3, 0)\end{aligned}$$

The third vertex is  $C(2, 5)$ ,  $C_1(0, -1)$ ,  $D(-1, 6)$ , or  $D_1(-3, 0)$ .

4. Find  $p$  and  $q$ , if one root of the equation  $x^2 + px + q = 0$  is 2, and the other root is three times the discriminant of the equation.

Solution: Let  $D$  be the discriminant; the roots are 2 and  $3D$ , so the equation is  $x^2 - (2 + 3D)x + 6D = 0$ , and

$$D = (2 + 3D)^2 - 24D, \quad 9D^2 - 13D + 4 = 0$$

so  $D = 1$  or  $D = \frac{4}{9}$ . If  $D = 1$ , then  $p = -5$ ,  $q = 6$ ; if  $D = \frac{4}{9}$ , then  $p = -\frac{10}{3}$ ,  $q = \frac{8}{3}$ .

5. In the triangle  $ABC$ ,  $AC = 10$ ,  $BC = 15$ , and  $\angle ACB = 60^\circ$ .

- a) What is the radius of the semicircle whose diameter lies on the side  $AB$  and which is tangent to the other two sides?  
b) What is the radius of the circle that is tangent to this semicircle and to the sides  $AC$  and  $BC$ ?

Solution: Let the radius of the semicircle be  $r$ , its center  $O$ , and the radius of the circle  $x$ , its center  $O_1$ .  $O$  is equally far from  $AC$  and  $BC$ , so it is on the bisector of the angle  $ACB$ , and so is  $O_1$ .

- a) The triangle  $ABC$  is made up of the triangles  $AOC$  and  $BOC$ , whose altitudes from  $O$  are  $r$ . So

$$\frac{10 \cdot 15 \cdot \sin 60^\circ}{2} = \frac{75\sqrt{3}}{2} = \frac{10r}{2} + \frac{15r}{2} = \frac{25r}{2}, \quad r = 3\sqrt{3}$$

- b) The bisector makes a  $30^\circ$  angle with  $AC$ , so  $O_1C = 2x$  and  $OC = 2r$ . If  $E$  is the point of tangency of the circle and the semicircle, then  $OC = OE + EC = r + x + 2x$ , so  $2r = r + 3x$ , and  $x = \frac{r}{3} = \sqrt{3}$ .

6. The difference of an arithmetic sequence is  $\frac{2}{3}$ , the sum of the first  $n$  terms is  $\frac{8}{3}$ , and the sum of the first  $(n + 3)$  terms is  $\frac{44}{3}$ . Find  $n$  and the first term of the sequence.

Solution:  $a_{n+1} + a_{n+2} + a_{n+3} = \frac{44}{3} - \frac{8}{3} = 12$ , so  $3a_{n+2} = 12$ ,  $a_{n+2} = 4$ , that is,  $a_1 + (n + 1) \cdot \frac{2}{3} = 4$ , and  $3a_1 + 2n = 10$ . Also

$$\frac{n}{2} \left( 2a_1 + (n - 1) \cdot \frac{2}{3} \right) = \frac{8}{3}, \quad n(6a_1 + 2n - 2) = 16$$

and substituting  $3a_1 = 10 - 2n$ :  $n^2 - 9n + 8 = 0$ , so  $n = 8$  or  $n = 1$ .

If  $n = 8$ :  $a_1 = -2$  (the terms are  $-2, -\frac{4}{3}, \dots$ , and indeed  $S_8 = \frac{8}{3}$ ,  $S_{11} = \frac{44}{3}$ ).

If  $n = 1$ :  $a_1 = \frac{8}{3}$  (the terms are  $\frac{8}{3}, \frac{10}{3}, 4, \frac{14}{3}$ , and indeed  $S_1 = \frac{8}{3}$ ,  $S_4 = \frac{44}{3}$ ).

(Note: the KoMaL solution says that  $n = 1$  does not satisfy the conditions, but as shown above, it does.)

7. Consider the function  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,

$$f(x) = \frac{(x+5)^2 - (x-5)^2}{(2x+5)^2 + (2x-5)^2}$$

Find the largest and smallest values of the function, and the values of  $x$  where they are taken.

Solution: The function is defined for every real  $x$ , and  $f(x) = \frac{20x}{8x^2 + 50} = \frac{10x}{4x^2 + 25}$ . It is odd:  $f(-x) = -f(x)$ .

If  $x > 0$ , then  $f(x) = \frac{10}{4x + \frac{25}{x}} > 0$ , and by the inequality between the arithmetic and geometric means

$$4x + \frac{25}{x} \geq 2\sqrt{4x \cdot \frac{25}{x}} = 20, \quad \text{so} \quad f(x) \leq \frac{10}{20} = \frac{1}{2}$$

with equality for  $4x = \frac{25}{x}$ , that is,  $x = \frac{5}{2}$ . If  $x < 0$ , then by symmetry  $f(x) \geq -\frac{1}{2}$ , with equality at  $x = -\frac{5}{2}$ . So the largest value is  $\frac{1}{2}$ , at  $x = \frac{5}{2}$ , and the smallest value is  $-\frac{1}{2}$ , at  $x = -\frac{5}{2}$ .

8. The diagonals  $AC$  and  $BD$  of the convex quadrilateral  $ABCD$  meet at  $K$ . The areas of the triangles  $ABK$ ,  $BCK$ ,  $CDK$ , and  $DAK$  are  $t_1$ ,  $t_2$ ,  $t_3$ , and  $t_4$ . Prove that the sides  $AB$  and  $DC$  of the quadrilateral are parallel if and only if the area  $T$  of the quadrilateral is

$$T = (\sqrt{t_1} + \sqrt{t_3})^2$$

Solution: The triangles with areas  $t_1$  and  $t_2$  have a common altitude (from  $B$  to  $AC$ ), and so do the triangles with areas  $t_4$  and  $t_3$  (from  $D$ ). So  $\frac{t_1}{t_2} = \frac{AK}{KC} = \frac{t_4}{t_3}$ , and  $t_1 t_3 = t_2 t_4$ .

If  $AB \parallel DC$ , then the triangles  $ABC$  and  $ABD$  have the same base  $AB$  and equal altitudes, so  $t_1 + t_2 = t_1 + t_4$ , that is,  $t_2 = t_4$ . Then  $t_1 t_3 = t_2^2$ ,  $\sqrt{t_1 t_3} = t_2$ , and

$$T = t_1 + t_2 + t_3 + t_4 = t_1 + t_3 + 2\sqrt{t_1 t_3} = (\sqrt{t_1} + \sqrt{t_3})^2$$

Conversely, if  $T = (\sqrt{t_1} + \sqrt{t_3})^2$ , then  $2\sqrt{t_1 t_3} = t_2 + t_4$ , and since  $t_1 t_3 = t_2 t_4$ ,  $t_2 + t_4 - 2\sqrt{t_2 t_4} = (\sqrt{t_2} - \sqrt{t_4})^2 = 0$ , so  $t_2 = t_4$ . So  $t_1 + t_2 = t_1 + t_4$ : the triangles  $ABC$  and  $ABD$  with the common side  $AB$  have equal areas, so their altitudes to  $AB$  are equal. The points  $C$  and  $D$  are on the same side of  $AB$  at equal distances from it, so  $AB \parallel CD$ .