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Pdf file: KoMaL, March 2004, page 134

In 1983, a new system of entrance examination was adopted. In order to be accepted to a university, the 3rd and 4th years of high school grades in mathematics, Hungarian language and literature, history, foreign language, physics, (biology, chemistry, geography, another foreign language - students choose from these) are counted toward university entrance performance.

The 'brought' points (i.e. the points comprised of final grades in high school in the subjects listed above) add up to a total of 60 points. In addition, students are assessed by written and oral examinations for a total of 60 points. So, there is a total of 120 points possible.

In Mathematics, the same exam serves as the GED and the university entrance exam. These problem sets consist of 8 problems, presented in order of difficulty (from easiest to most difficult).

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present all problems.

1. Solve the following equations.

$$\text{a) } \frac{x^3 - 4x}{2 - x} = 0 \quad \text{b) } \sqrt{\frac{x^3 - 4x}{2 - x}} = \frac{\sqrt{3}}{2} \quad \text{c) } \lg \frac{x^3 - 4x}{2 - x} = 0$$

Solution: In all three equations $x \neq 2$, and then $\frac{x^3 - 4x}{2 - x} = \frac{x(x - 2)(x + 2)}{2 - x} = -x(x + 2)$.

a) $-x(x + 2) = 0$, so $x_1 = -2$ and $x_2 = 0$.

b) The square root requires $-x(x + 2) \geq 0$, that is, $-2 \leq x \leq 0$. Squaring: $-x^2 - 2x = \frac{3}{4}$, that is, $x^2 + 2x + \frac{3}{4} = 0$, so $x_1 = -\frac{3}{2}$ and $x_2 = -\frac{1}{2}$, and both are solutions.

c) The logarithm requires $-x(x + 2) > 0$, that is, $-2 < x < 0$. The equation says $-x^2 - 2x = 10^0 = 1$, that is, $(x + 1)^2 = 0$, so the only solution is $x = -1$.

(lg denotes the base 10 logarithm.)

2. Prove that if $a + 1$ is divisible by 6, then $n(an^2 + 1)$ is divisible by 6 for every integer n .

Solution: If $a + 1$ is divisible by 6, then $a = 6k - 1$ for some integer k , and

$$n(an^2 + 1) = n((6k - 1)n^2 + 1) = 6kn^3 - (n - 1)n(n + 1)$$

The first term is a multiple of 6, and the second is the product of three consecutive integers, so it is also divisible by 6.

3. In a circle of radius 5 cm, the chord AB is equal to the side of the regular octagon inscribed in the circle, and the chord BC is equal to the side of the regular hexagon inscribed in the circle. How long is the segment AC ?

Solution: Let O be the center. The central angle of AB is 45° , and the central angle of BC is 60° . A and C can be on opposite sides of B or on the same side, so the central angle of AC is $45^\circ + 60^\circ = 105^\circ$ or $60^\circ - 45^\circ = 15^\circ$. A chord with central angle φ is $2 \cdot 5 \sin \frac{\varphi}{2}$ long, so

$$AC = 10 \sin 52.5^\circ \approx 7.9 \text{ cm} \quad \text{or} \quad AC = 10 \sin 7.5^\circ \approx 1.3 \text{ cm}$$

(The same values can be obtained with the Law of Cosines: $AB = \sqrt{50 - 25\sqrt{2}} \approx 3.8268 \text{ cm}$, $BC = 5 \text{ cm}$, and the angle ABC is $67.5^\circ + 60^\circ = 127.5^\circ$ or $67.5^\circ - 60^\circ = 7.5^\circ$.)

4. For which values of the parameter a does the equation

$$a^2(x-1) = 4(ax-x-1)$$

have an integer root?

Solution: The equation can be rearranged to $(a-2)^2 \cdot x = a^2 - 4$.

If $a = 2$: the equation is $0 \cdot x = 0$, so every real number is a solution, and in particular there are integer roots.

If $a \neq 2$:

$$x = \frac{(a-2)(a+2)}{(a-2)^2} = \frac{a+2}{a-2} = 1 + \frac{4}{a-2}$$

If a is an integer (this is the case the KoMaL solution considers), x is an integer exactly when $a-2$ is a divisor of 4, so $a = -2, 0, 1, 3, 4, 6$; together with $a = 2$ there are seven such integers: $a \in \{-2, 0, 1, 2, 3, 4, 6\}$.

(If a can be any real number, then for every integer $x \neq 1$ we can solve $x = \frac{a+2}{a-2}$ for a : $a = \frac{2(x+1)}{x-1}$. So in that case the answer is $a = 2$ and all numbers of the form $a = \frac{2(x+1)}{x-1}$, where $x \neq 1$ is an integer; for example $a = \frac{10}{3}$ gives the root $x = 4$.)

5. Two vertices of the triangle ABC of area 13 are $A(3, -1)$ and $B(-1, 5)$. Find the coordinates of the vertex C so that the perimeter of the triangle is as small as possible.

Solution: $AB = 2\sqrt{13}$, and since the area is 13, the altitude from C is $m_c = \sqrt{13}$. So C is on one of the two lines f_1, f_2 parallel to AB at distance $\sqrt{13}$. For a fixed line f_1 , the sum $AC + CB$ is smallest when C is the intersection of f_1 with $A'B$, where A' is the reflection of A in f_1 (then $AC + CB = A'C + CB$); since $f_1 \parallel AB$, this point is on the perpendicular bisector of AB , so the triangle is isosceles.

Rotating the segment AB by 90° around its midpoint $F(1, 2)$, we get the segment KL with $K(-2, 0)$ and $L(4, 4)$, and $FK = FL = \sqrt{13}$. So the possible coordinates of C are $(-2, 0)$ and $(4, 4)$.

6. On two neighboring sides of a rectangle with perimeter 60 cm we draw semicircles outward, with the sides as diameters. How should we choose the sides of the rectangle so that the resulting figure

- a) has the smallest perimeter; b) has the largest area?

Solution: Let one side of the rectangle be x ; then the other side is $30 - x$.

- a) The perimeter of the figure is

$$k(x) = 30 + \frac{x}{2} \cdot \pi + \frac{30-x}{2} \cdot \pi = 30 + 15\pi$$

so the perimeter does not depend on x : it is the same for every such rectangle.

- b) The area of the figure is

$$t(x) = x(30-x) + \frac{1}{2} \left(\frac{x}{2}\right)^2 \pi + \frac{1}{2} \left(\frac{30-x}{2}\right)^2 \pi = \left(\frac{\pi}{4} - 1\right)x^2 + \frac{60-15\pi}{2}x + \frac{225\pi}{2}$$

The leading coefficient is negative, so the maximum is at $x = -\frac{b}{2a} = \frac{15\pi - 60}{\pi - 4} = 15$. The area is largest if the rectangle is a square with side 15 cm.

7. The angles α and β of a triangle satisfy

$$(1 + \cot \alpha)(1 + \cot \beta) = 2 \cot \alpha \cot \beta$$

Find the third angle of the triangle.

Solution: Multiplying both sides by $\sin \alpha \sin \beta$:

$$(\sin \alpha + \cos \alpha)(\sin \beta + \cos \beta) = 2 \cos \alpha \cos \beta, \quad \sin \alpha \cos \beta + \cos \alpha \sin \beta = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

that is, $\sin(\alpha + \beta) = \cos(\alpha + \beta)$. Since $0^\circ < \alpha + \beta < 180^\circ$, $\sin(\alpha + \beta) \neq 0$, so $\cot(\alpha + \beta) = 1$, and $\alpha + \beta = 45^\circ$. The third angle is 135° .

8. We join the vertex A of the convex pentagon $ABCDE$ with the midpoints of the diagonal BE and of the sides BC , CD , and DE . On these four segments, the points of trisection farther from A determine a quadrilateral whose area is one fifth of the area of the pentagon. What fraction of the area of the pentagon is the area of the triangle ABE ?

Solution: Let F_2, F_3, F_4 be the midpoints of BC, CD, DE , and F the midpoint of BE ; we join A with F_2, F_3, F_4 , and F . The four points of trisection farther from A are the images of F_2, F_3, F_4, F under the dilation with center A and ratio $\frac{2}{3}$.

$F_2F_3F_4F$ is the quadrilateral of the midpoints of the sides of the quadrilateral $BCDE$, so it is a parallelogram with half the area T of $BCDE$. The dilation multiplies areas by $\frac{4}{9}$, so the quadrilateral in the problem has area $\frac{4}{9} \cdot \frac{T}{2} = \frac{2}{9}T$. Let the area of the triangle ABE be μT . Then

$$\frac{1}{5} = \frac{\frac{2}{9}T}{\mu T + T} = \frac{2}{9\mu + 9}, \quad \mu = \frac{1}{9}$$

So the area of the triangle ABE is $\frac{T}{9}$, and the pentagon has area $\frac{10T}{9}$: the triangle ABE is one tenth of the pentagon.