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translated from: KoMaL, November 2004, pages 462-463

Pdf file: KoMaL, November 2004, pages 462-463

From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. Solve the equation

$$x \cdot \left(\frac{x-3}{3} \right)^2 + \frac{1}{3} (x-2)(x+2) = \frac{1+(x-1)^3}{9}$$

Solution: Multiplying by 9: $x(x-3)^2 + 3(x^2-4) = 1+(x-1)^3$, that is, $x^3 - 3x^2 + 9x - 12 = x^3 - 3x^2 + 3x$. This is the linear equation $6x = 12$, and the solution is $x = 2$.

2. We roll a die 6 times and record the results.

- How many different sequences can we get?
- How many different sequences can we get, if there is a 1 in the first place and nowhere else?
- What is the probability that the number in the first place is different from all the others?

Solution: a) $6^6 = 46\,656$.

b) The first place is fixed, and each of the other five places can be any of the other five numbers: $5^5 = 3125$.

c) The first number can be any of the 6, and then each of the other five places can be any of the five other numbers, so there are $6 \cdot 5^5$ favorable cases, and

$$P = \frac{6 \cdot 5^5}{6^6} = \frac{5^5}{6^5} \approx 0.402$$

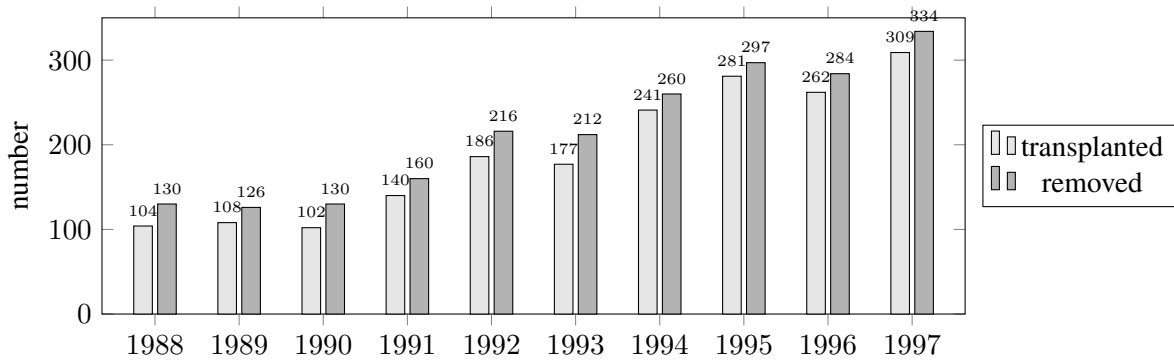
3. A math teacher says that she composed the test well if the average of the grades is between e and π ($e = 2.718\dots$ and $\pi = 3.141\dots$). In a class of 35, she has not yet looked at the work of the best student, and at that point the average was 2.68. How could the best student's test turn out, if the teacher could say about herself that she composed the test well? (Grades are integers from 1 to 5.)

Solution: The average of the 34 grades is 2.68 after rounding, so their sum S satisfies $2.675 \cdot 34 = 90.95 \leq S < 2.685 \cdot 34 = 91.29$, that is, $S = 91$. If the best student's grade is $n \in \{1, 2, 3, 4, 5\}$, we need

$$2.718 < \frac{91+n}{35} < 3.141$$

and of the five possible values only $n = 5$ works. The best student got a 5, and then the average was $\frac{96}{35} \approx 2.74$.

4. Consider the following diagram. It shows the numbers of transplanted and removed kidneys over a 10-year period.



- a) In which year was the largest percentage of the removed kidneys transplanted?
 b) Show in a diagram the ratio of transplanted to removed kidneys in the first three and the last three years.
 c) Can any conclusion be drawn from the resulting diagram?

Solution: a) The ratio of transplanted to removed kidneys in each year:

1988	1989	1990	1991	1992	1993	1994	1995	1996	1997
80%	85.7%	78.5%	87.5%	86.1%	83.5%	92.7%	94.6%	92.3%	92.5%

The largest percentage of the removed kidneys was transplanted in 1995.

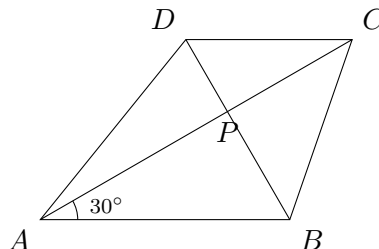
- b) For example, with a bar chart whose vertical axis is divided into tenths, showing the ratios 0.80, 0.86, 0.79 for 1988–1990 and 0.92, 0.95, 0.93 for 1995–1997.
 c) The ratio is clearly higher in the last three years of the period than in the first three. (Probably the equipment of hospitals for removing, transporting, and storing kidneys improved.)

Part II

5. One diagonal of a trapezoid makes a 30° angle with the base. The two diagonals are perpendicular.

- a) How long are the two diagonals, if the parallel sides are 6 and 4 units long?
 b) What is the area of the trapezoid? c) Find the perimeter of the trapezoid.

Solution: Let the trapezoid be $ABCD$ with $AB = 6$, $CD = 4$, let the diagonal AC make the 30° angle with AB , and let the diagonals meet at P .



- a) $\angle BAC = \angle DCA$ (alternate angles), so PAB and PCD are right triangles with angles 30° and 60° . So $BP = 3$, $AP = 3\sqrt{3}$, and $DP = 2$, $CP = 2\sqrt{3}$. The diagonals are $AC = 5\sqrt{3} \approx 8.66$ and $BD = 5$.

- b) The diagonals are perpendicular, so the area is half their product: $t = \frac{5 \cdot 5\sqrt{3}}{2} = \frac{25\sqrt{3}}{2} \approx 21.65$.

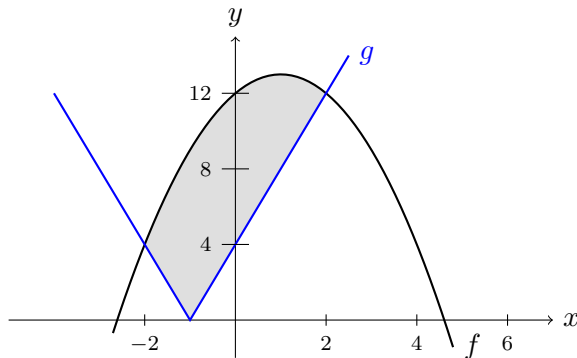
- c) By the Pythagorean Theorem the legs are $AD = \sqrt{AP^2 + DP^2} = \sqrt{31} \approx 5.57$ and $BC = \sqrt{BP^2 + CP^2} = \sqrt{21} \approx 4.58$, so the perimeter is $10 + \sqrt{31} + \sqrt{21} \approx 20.15$.

6. Consider the functions $f(x) = 13 - (x - 1)^2$ and $g(x) = 4|x + 1|$ defined on the real numbers.

a) Solve the inequality $f(x) \geq g(x)$ graphically.

b) What is the area of the region bounded by the graphs of f and g ?

Solution: a) For $x \geq -1$, $f(x) = g(x)$ means $x^2 + 2x - 8 = 0$, so $x = 2$; for $x < -1$ it means $x^2 - 6x - 16 = 0$, so $x = -2$. The graph:



From the graph, the solution is $-2 \leq x \leq 2$.

b) The area under the graph of f on $[-2, 2]$ is

$$\int_{-2}^2 (-x^2 + 2x + 12) dx = \left[-\frac{x^3}{3} + x^2 + 12x \right]_{-2}^2 = \frac{128}{3}$$

The area under the graph of g is the sum of the areas of two right triangles: $\frac{1 \cdot 4}{2} + \frac{3 \cdot 12}{2} = 20$. The area of the region is $\frac{128}{3} - 20 = \frac{68}{3} \approx 22.67$.

7. Prove that the difference of the roots of the equation

$$5x^2 - 2(5k + 3)x + 5k^2 + 6k + 1 = 0$$

is the same for every value of k .

Solution: One quarter of the discriminant is $(5k + 3)^2 - 5(5k^2 + 6k + 1) = 4$, so

$$x_{1,2} = \frac{(5k + 3) \pm 2}{5}, \quad \text{that is,} \quad x_1 = k + 1, \quad x_2 = k + 0.2$$

The difference of the roots is 0.8 for every k .

8. In the triangle ABC , $\beta - \gamma = 90^\circ$. Prove that the line of the altitude from the vertex A is also tangent to the circle circumscribed around the triangle.

Solution: Since $\beta > 90^\circ$, the foot T of the altitude from A is on the extension of CB beyond B . In the right triangle ATB the angle at B is $180^\circ - \beta$, so

$$\angle TAB = 90^\circ - (180^\circ - \beta) = \beta - 90^\circ = \gamma$$

So the angle between the line AT and the chord AB is equal to the inscribed angle ACB over the arc AB , on the other side of the chord. By the converse of the tangent-chord angle theorem, the line AT is tangent to the circumscribed circle at A .

9. Among the isosceles triangles with perimeter 50 cm, choose the one for which the sum of the areas of the squares drawn on its sides is as small as possible.

Solution: Let the legs be b ; then the base is $a = 50 - 2b$, where $12.5 < b < 25$ (triangle inequality). The sum of the areas of the squares is

$$t(b) = 2b^2 + (50 - 2b)^2 = 6b^2 - 200b + 2500 = 6\left(b - \frac{50}{3}\right)^2 + \frac{2500}{3}$$

This is smallest for $b = \frac{50}{3}$, which is in the allowed interval, and then $a = \frac{50}{3}$ too. The triangle is equilateral with side $\frac{50}{3}$ cm, and the sum of the areas is $\frac{2500}{3} \approx 833.3 \text{ cm}^2$.