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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. Find the coordinates of the vertices of the square whose diagonals are parallel to the coordinate axes and three of whose vertices are on the parabola $x^2 - 6x - 5y + 4 = 0$.

- a) What is the area of the circle inscribed in the square?
b) How many lattice points does the closed disk have?

(11 points)

Solution: The parabola is $\frac{(x-3)^2}{5} = y + 1$, with vertex $C(3, -1)$ and axis $x = 3$. The vertices of a square with axis-parallel diagonals are $(p, q \pm d)$ and $(p \pm d, q)$. Two points with the same x -coordinate cannot both be on the parabola, so $(p - d, q)$ and $(p + d, q)$ are on it, and by symmetry $p = 3$. The third vertex on the parabola is $(3, q - d)$ or $(3, q + d)$, so it is the vertex $C(3, -1)$ of the parabola (the upper one would need $q + d = -1$, which is impossible). The sides of the square make 45° angles with the axes, so the side through C is on the line $y = x - 4$, which meets the parabola again at $D(8, 4)$. So the vertices are

$$A(3, 9), \quad B(-2, 4), \quad C(3, -1), \quad D(8, 4)$$

- a) The diagonal of the square is 10, so its side is $5\sqrt{2}$, which is the diameter of the inscribed circle. Its area is $T = \frac{25\pi}{2} \approx 39.3$, and its equation is $(x-3)^2 + (y-4)^2 = \frac{25}{2}$.

- b) We count the integer pairs $(u, v) = (x-3, y-4)$ with $u^2 + v^2 \leq 12.5$, that is, $u^2 + v^2 \leq 12$. On the lines $u = 0$ and $v = 0$ there are 13 such points ($|u| \leq 3$ or $|v| \leq 3$), and in each quadrant there are 6 more: $(1, 1), (1, 2), (2, 1), (2, 2), (1, 3), (3, 1)$. So the closed disk has $13 + 4 \cdot 6 = 37$ lattice points.

2. Solve the following equation over the real numbers.

$$1 - x^2 - x^4 = 4^{\sin^2 x}$$

(12 points)

Solution: Since $0 \leq \sin^2 x \leq 1$, the right-hand side satisfies $1 \leq 4^{\sin^2 x} \leq 4$. The left-hand side is at most 1, and it is 1 only for $x = 0$. For $x = 0$ both sides are 1, so $x = 0$ is the only solution.

3. The following table shows the change of the consumer price index in some countries between 1995 and 1999:

Country	change compared to the previous year, %				
	1995	1996	1997	1998	1999
Austria	2.3	1.5	1.3	1	0.6
France	1.7	2	1.2	0.8	0.6
Poland	27.8	19.9	14.8	11.6	7.3
Hungary	28.2	23.6	18.3	14.3	10
Great Britain	3.4	2.4	3.2	3.4	1.6
Germany	1.8	1.4	1.9	1	0.6

- a) Explain what the numbers 1 and 0.6 in the last two columns for Austria mean.
- b) If someone hid 34 000 forints in their mattress in 1994, how many forints did its purchasing power correspond to in 1999?
- c) In 1999, when her grandchild graduates, Grandma wants to give 200 000 forints as a graduation present. How much money did she put in 1994, upon hearing the news of the successful entrance exam, into a bank that, as it turned out later, paid interest exactly equal to the consumer price index between 1994 and 1999?

(14 points)

Solution: a) In Austria, consumer prices in 1998 were on average 1.01 times, and in 1999 1.006 times, those of the previous year.

- b) The 1994 forint was worth only $\frac{1}{1.282}$ of its value in 1995, and so on, so the purchasing power of the 34 000 forints in 1999 corresponded to

$$\frac{34\,000}{1.282 \cdot 1.236 \cdot 1.183 \cdot 1.143 \cdot 1.10} \approx 14\,426 \text{ forints of 1994}$$

- c) By the same computation, $\frac{200\,000}{1.282 \cdot 1.236 \cdot 1.183 \cdot 1.143 \cdot 1.10} \approx 84\,859$, so Grandma put about 85 000 forints in the bank in 1994.

4. The base of the pyramid with apex D is the equilateral triangle ABC . The base edges are 18 cm, and the lateral edges are $6\sqrt{6}$ cm long. We pass a plane parallel to the edge AD through the midpoints of the edges AB and AC .
- a) Find the area of the cross section.
- b) What angle does the cutting plane make with the plane of the base?

(14 points)

Solution: Let F_1 and F_2 be the midpoints of AB and AC . The plane through F_1F_2 parallel to AD meets the faces ABD and ACD in midlines parallel to AD , which end at the midpoints P and Q of BD and CD . So the cross section F_1F_2QP is a parallelogram, with sides $F_1F_2 = \frac{BC}{2} = 9$ cm and $F_1P = \frac{AD}{2} = 3\sqrt{6}$ cm, and by symmetry its diagonals are equal, so it is a rectangle.

- a) The area of the rectangle is $t = 9 \cdot 3\sqrt{6} = 27\sqrt{6} \approx 66.1$ cm².

- b) The cross section contains the line F_1F_2 , which is perpendicular to the plane through AD and the altitude of the pyramid, and it is parallel to AD , so it makes the same angle with the base as AD . The projection of D onto the base is the center of the triangle ABC , so the projection of AD is the radius of the circumscribed circle, $6\sqrt{3}$ cm, and

$$\cos \alpha = \frac{6\sqrt{3}}{6\sqrt{6}} = \frac{\sqrt{2}}{2}, \quad \alpha = 45^\circ$$

Part II

5. In a group of 16 students, the chemistry average was 3.81 (rounded to two decimal places). We know that nobody failed (grades are from 1 to 5, and 1 is failing).
- At most how many students could have received a 2?
 - Is it certain that someone received a 5?
 - In how many ways could there have been exactly 11 grades of 4?
 - Is it true that if the mode is 4, then the median is also 4?

(16 points)

Solution: The sum x of the grades satisfies $3.805 \cdot 16 \leq x < 3.815 \cdot 16$, that is, $60.88 \leq x < 61.04$, so $x = 61$.

- At most six: with seven grades of 2, the remaining 9 grades would have to add up to 47, which is impossible ($9 \cdot 5 = 45$). Six 2s, one 4, and nine 5s satisfy all the conditions.
- No: for example, three 3s and thirteen 4s also give 61.
- With eleven 4s, the remaining 5 grades add up to 17. They cannot all be 3 or less (at most 15), so there is at least one 5. With one 5, the other four are 3s; with two 5s, the other three are one 3 and two 2s; three 5s are impossible (the other two would add up to 2). So there are two possibilities.
- True. If 4 is the mode, then there are at least five 4s: otherwise every grade occurs at most 4 times, so each of 2, 3, 4, 5 occurs exactly 4 times, the sum is $56 \neq 61$, and there is no single mode. Let a be the number of grades below 4, $f \geq 5$ the number of 4s, and b the number of 5s ($a + f + b = 16$). The median is 4 exactly when the 8th and 9th grades in increasing order are both 4, that is, $a \leq 7$ and $b \leq 7$.

If $a \geq 8$, then $b \leq 3$, and the sum is at most $3a + 4f + 5b \leq 3 \cdot 8 + 4 \cdot 5 + 5 \cdot 3 = 59 < 61$. If $b \geq 8$, then $a \leq 3$, and the sum is at least $2a + 4f + 5b \geq 2 \cdot 3 + 4 \cdot 5 + 5 \cdot 8 = 66 > 61$. Both are impossible, so the median is 4.

6. The radius of the inscribed circle, the radius of the circumscribed circle, and the perimeter of a right triangle are three consecutive terms of a geometric sequence. Prove that then the sum of the tangents of the two acute angles is 4.

(16 points)

Solution: Let the radius of the inscribed circle be r , of the circumscribed circle R , the legs a and b , and the hypotenuse c . By the equality of tangent segments $r = \frac{a+b-c}{2}$, and by Thales' theorem $R = \frac{c}{2}$. So

$$\left(\frac{c}{2}\right)^2 = \frac{a+b-c}{2} \cdot (a+b+c) = \frac{(a+b)^2 - c^2}{2} = ab$$

(using $c^2 = a^2 + b^2$), that is, $a^2 + b^2 = 4ab$, or $a^2 - 4ab + b^2 = 0$. Dividing by b^2 , $\tan \alpha = \frac{a}{b}$ satisfies $t^2 - 4t + 1 = 0$,

so $\tan \alpha = 2 \pm \sqrt{3}$. These two numbers are reciprocals of each other, so one is $\tan \alpha$ and the other is $\tan \beta = \frac{1}{\tan \alpha}$,

and their sum is 4. (More directly: $\tan \alpha + \tan \beta = \frac{a}{b} + \frac{b}{a} = \frac{a^2 + b^2}{ab} = 4$.)

7. The distance of a water tower (P) from the village A has to be determined with the help of a map of scale 1 : 250 000. For this, the distances of two villages (B and C) from A are known from the map: $AB = 17.2$ cm and $AC = 20.0$ cm. We also measured that

$$\angle APB = 83^\circ, \quad \angle APC = 130^\circ, \quad \angle BAC = 65^\circ$$

What is the distance AP in reality?

(16 points)

Solution: Let $\varphi = \angle PAB$; then $\angle PAC = 65^\circ - \varphi$ (P is inside the angle BAC , and $ABPC$ is a concave quadrilateral). In the triangle ABP the angle at B is $180^\circ - 83^\circ - \varphi = 97^\circ - \varphi$, and in the triangle ACP the angle at C is $180^\circ - 130^\circ - (65^\circ - \varphi) = \varphi - 15^\circ$. By the Law of Sines

$$AP = \frac{AB \sin(97^\circ - \varphi)}{\sin 83^\circ} = \frac{AC \sin(\varphi - 15^\circ)}{\sin 130^\circ}$$

Expanding the sines, this gives an equation for $\tan \varphi$:

$$\tan \varphi = \frac{N}{M}, \quad \text{where } N = \frac{17.2 \sin 97^\circ}{\sin 83^\circ} + \frac{20 \sin 15^\circ}{\sin 130^\circ}, \quad M = \frac{17.2 \cos 97^\circ}{\sin 83^\circ} + \frac{20 \cos 15^\circ}{\sin 130^\circ}$$

and only the acute solution makes sense: $\varphi \approx 46.04^\circ$. Then $AP \approx 13.46$ cm on the map. Since 1 cm on the map is 2.5 km, the real distance is about 33.65 km.

8. Find the cubic function with the following properties:

- it touches the curve $x^2 + y = 4$ from inside at $x = 0$;
- the two curves meet at $x = 2$;
- the area enclosed by the two curves on the interval $[0, 2]$ is $\frac{4}{3}$.

Describe the behavior of the cubic function.

(16 points)

Solution: Let the function be $x \mapsto ax^3 + bx^2 + cx + d$. It touches the parabola $y = 4 - x^2$ at $x = 0$, so it passes through $(0, 4)$, $d = 4$, and its derivative at 0 is 0, $c = 0$. The two curves meet at $x = 2$, so $8a + 4b + 4 = 0$. The cubic is below the parabola on $[0, 2]$, so

$$\int_0^2 (4 - x^2) dx - \int_0^2 (ax^3 + bx^2 + 4) dx = -\frac{8}{3} - 4a - \frac{8}{3}b = \frac{4}{3}, \quad 3a + 2b = -3$$

From $2a + b = -1$ and $3a + 2b = -3$: $a = 1$, $b = -3$. The function is

$$x \mapsto x^3 - 3x^2 + 4 = (x + 1)(x - 2)^2$$

Its derivative is $3x^2 - 6x = 3x(x - 2)$, so it is increasing on $(-\infty, 0]$, has a local maximum 4 at $x = 0$, is decreasing on $[0, 2]$, has a local minimum 0 at $x = 2$, and is increasing on $[2, \infty)$. Its zeros are -1 and 2 (a double zero).

9. A group of 9 people gets on a suburban train of three cars, but in the big crowd every member of the group only cares about getting on one of the cars, not about which cars the others got on.
- What is the probability that 3 members of the group got on each of the three cars?
 - What is the probability that nobody from the group got on at least one of the three cars?
 - What is the probability that at most one person from the group got on at least one of the three cars?

(16 points)

Solution: Each person chooses one of the cars 1, 2, 3, so there are $3^9 = 19\,683$ equally likely outcomes.

a) The number of outcomes in which each car gets 3 people is $\frac{9!}{3!3!3!} = 1680$, so $p = \frac{1680}{3^9} \approx 0.085$.

b) The outcomes using only two given cars number 2^9 , and there are $\binom{3}{2} = 3$ pairs of cars, but this way the 3 outcomes in which everybody got on the same car are counted twice. So

$$p = \frac{3 \cdot 2^9 - 3}{3^9} = \frac{1533}{19\,683} \approx 0.0779$$

c) The bad outcomes are those in which at least two people got on each car, that is, the numbers of people in the cars are 2, 2, 5, or 2, 3, 4, or 3, 3, 3 (in some order); the number of such outcomes for a given order is $\frac{9!}{2!2!5!} = 756$, $\frac{9!}{2!3!4!} = 1260$, and $\frac{9!}{3!3!3!} = 1680$. So

$$p = 1 - \frac{3 \cdot 756 + 6 \cdot 1260 + 1680}{3^9} = 1 - \frac{11\,508}{19\,683} \approx 0.4153$$