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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

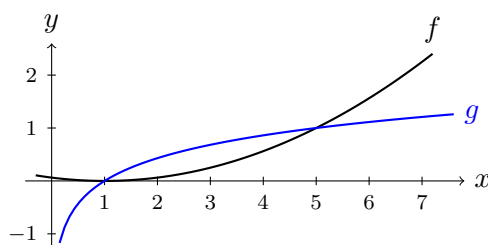
Part I

1. Solve the following inequality graphically over the real numbers.

$$\frac{(x-1)^2}{16} \leq \log_5 x$$

(11 points)

Solution: We graph the functions $f(x) = \frac{(x-1)^2}{16}$ and $g(x) = \log_5 x$. From the graph (and by substitution) the two curves meet at $x = 1$ and $x = 5$: $f(1) = g(1) = 0$ and $f(5) = g(5) = 1$.



On the interval $[1, 5]$ the graph of g is on or above the graph of f , and outside it below. (For $x \leq 0$, g is not defined.) The solution is $1 \leq x \leq 5$.

2. The face of a clock is an equilateral triangle with side 20 cm. The hands are fixed at the center of the triangle so that at 12 o'clock they point toward one of the vertices.
 - a) How long can the minute hand be, if it never reaches beyond the face of the clock?
 - b) At 3 o'clock, what is the area of the part of the clock face cut off by the two rays determined by the two hands?

(12 points)

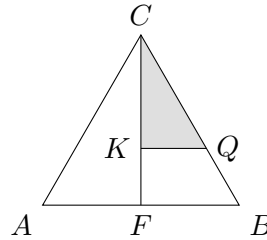
Solution: a) The minute hand turns all the way around, so it can be at most as long as the radius of the circle inscribed in the triangle. The area of the triangle is $t = \frac{a^2\sqrt{3}}{4} = 100\sqrt{3} \text{ cm}^2$, and half its perimeter is $s = 30 \text{ cm}$, so by $t = rs$

$$r = \frac{100\sqrt{3}}{30} = \frac{10\sqrt{3}}{3} \approx 5.77 \text{ cm}$$

The minute hand can be at most $\frac{10\sqrt{3}}{3}$ cm long.

b) Let the center be K , the vertex at 12 o'clock C , the midpoint of the opposite side F , and let the hour hand (pointing to 3, perpendicular to KC) meet the side BC at Q . The right triangles CKQ and CFB are similar (they have a common angle at C), and since K is the centroid, $CK : CF = 2 : 3$. So

$$t_{CKQ} = \left(\frac{2}{3}\right)^2 \cdot t_{CFB} = \frac{4}{9} \cdot \frac{1}{2} \cdot 100\sqrt{3} = \frac{200\sqrt{3}}{9} \approx 38.5 \text{ cm}^2$$



3. A brochure on healthy eating says, among other things, that cholesterol is found in foods of animal origin. The following table gives the cholesterol content of some foods (in mg per 100 g):

1. Brain	3000	10. Lean pork	68
2. Pork liver	460	11. Pressed ham	45
3. Beef liver	305	12. Poultry	38
4. Liver spread	224	13. Smoked pork loin	36
5. Cracklings	155	14. Butter	230
6. Winter salami	150	15. Emmental cheese	135
7. Smoked bacon	140	16. Cream cheese	83
8. Pork neck	95	17. Breast milk	20
9. Beef	75	18. Cow's milk	10

- In how many ways can three of the foods in the table be chosen?
- If a bar chart were made to compare the cholesterol content of brain, pork liver, beef liver, cracklings, and smoked bacon from the table, in how many different orders could the chosen foods appear?
- The names of the 18 foods in the table are written on identical slips, and three are chosen at random. What is the probability that the sum of the numbers written next to them is less than 70?
- The names of the 18 foods are written on identical slips, and three are chosen at random. What is the probability that the sum of the numbers written next to them is less than 3000?

(14 points)

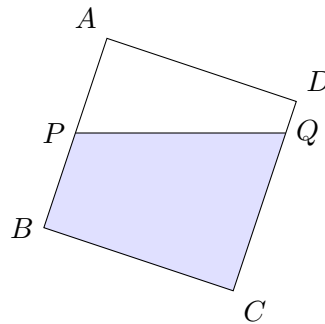
Solution: a) $\binom{18}{3} = \frac{18 \cdot 17 \cdot 16}{1 \cdot 2 \cdot 3} = 816$ ways.

b) The five foods can be listed in $5! = 120$ different orders.

c) Only the six foods with cholesterol content below 70 matter: 68, 45, 38, 36, 20, 10. The sum of three of them is less than 70 in exactly two cases: $38 + 20 + 10 = 68$ and $36 + 20 + 10 = 66$. All choices are equally likely, so the probability is $\frac{2}{816} = \frac{1}{408}$.

d) The only value of at least 1000 is 3000 (brain), and every other value is less than 1000. So the sum is less than 3000 exactly when brain is not chosen, in $\binom{17}{3} = 680$ cases. The probability is $\frac{680}{816} = \frac{5}{6}$.

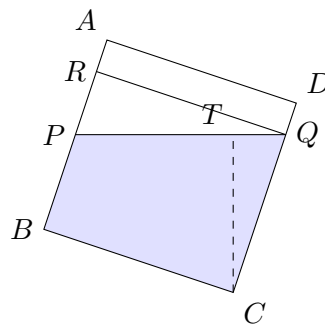
4. A cube-shaped container with edge 12 cm was filled with liquid up to $\frac{2}{3}$ of its height, and then tilted a little along one of its edges. The figure shows the cross section of the container with the level of the liquid in it.



We know that the point P bisects the segment AB . How high is the liquid above the vertex C ?

(14 points)

Solution: The cross section of the liquid is the trapezoid $PBCQ$ ($PB \parallel QC$), and its area is $\frac{2}{3} \cdot 12^2 = 96 \text{ cm}^2$. Since $PB = 6 \text{ cm}$ and $BC = 12 \text{ cm}$, $\frac{6 + QC}{2} \cdot 12 = 96$, so $QC = 10 \text{ cm}$. Let R be the point of AB with $RQ \parallel BC$, and T the foot of the perpendicular from C to the line PQ .



Then $PR = 4 \text{ cm}$, $RQ = 12 \text{ cm}$, and $QC = 10 \text{ cm}$. The right triangles PQR and QCT are similar, since their acute angles at Q and C are equal (angles with perpendicular sides). Since $PR : RQ = 4 : 12 = 1 : 3$, also $QT : TC = 1 : 3$, that is, $QT = \frac{TC}{3}$. By the Pythagorean Theorem in the triangle QTC :

$$\left(\frac{TC}{3}\right)^2 + TC^2 = 100, \quad TC = 3\sqrt{10} \approx 9.5 \text{ cm}$$

So the liquid is about 9.5 cm high above the vertex C .

Part II

5. Solve the following equation over the pairs of real numbers.

$$\left(\frac{1}{2} + \sin x\right)^2 + (y^2 + \cos x)^2 = 0$$

(16 points)

Solution: The sum of two nonnegative numbers is 0 only if both are 0. So $\sin x = -\frac{1}{2}$, that is, $x = -\frac{\pi}{6} + 2k\pi$ or $x = \frac{7\pi}{6} + 2k\pi$, and $y^2 + \cos x = 0$. For $x = -\frac{\pi}{6} + 2k\pi$, $\cos x > 0$, so $y^2 + \cos x = 0$ is impossible. For $x = \frac{7\pi}{6} + 2k\pi$, $\cos x = -\frac{\sqrt{3}}{2}$, so $y^2 = \frac{\sqrt{3}}{2}$. The solutions are

$$x = \frac{7\pi}{6} + 2k\pi, \quad y = \pm \sqrt{\frac{\sqrt{3}}{2}} \approx \pm 0.93, \quad k \text{ is an integer}$$

6. $f(x)$ is a quadratic function, and $g(x)$ is an inverse proportionality. We know that $f(1) = g(1) = 6$, $f(2) = g(2) = 3$, $f(3) = g(3) = 2$.

a) Is there any other x for which $f(x) = g(x)$? b) Find $g(12)$. c) Find $f(15)$.

d) Is there a real number x for which $f(x)$ is equal to the reciprocal of $g(x)$?

(16 points)

Solution: Let $f(x) = ax^2 + bx + c$. From $a + b + c = 6$, $4a + 2b + c = 3$, $9a + 3b + c = 2$ we get $a = 1$, $b = -6$, $c = 11$, so $f(x) = x^2 - 6x + 11$. Let $g(x) = \frac{d}{x}$; $g(1) = d = 6$, so $g(x) = \frac{6}{x}$, and the other two conditions also hold.

a) $x^2 - 6x + 11 = \frac{6}{x}$ ($x \neq 0$) means $x^3 - 6x^2 + 11x - 6 = 0$. Its roots are 1, 2, and 3, and a cubic equation has at most three roots, so there is no other such x .

b) $g(12) = \frac{6}{12} = \frac{1}{2}$. c) $f(15) = 15^2 - 6 \cdot 15 + 11 = 146$.

d) $x^2 - 6x + 11 = \frac{x}{6}$ means $6x^2 - 37x + 66 = 0$, whose discriminant $37^2 - 24 \cdot 66 = -215$ is negative. So there is no such x .

7. Let $N = 9 \cdot 2^{3n} + 64 \cdot 3^{4n}$.

a) Prove that there is no natural number n for which N is an odd multiple of 5.

b) Prove that N is divisible by 73 for every natural number n .

(16 points)

Solution: a) If $n = 0$, then $N = 73$. Otherwise N is the sum of two even numbers, so it is even, and it cannot be an odd multiple of 5.

b) $N = 9 \cdot 8^n + 64 \cdot 81^n = 64(81^n - 8^n) + 73 \cdot 8^n$. Since

$$81^n - 8^n = (81 - 8)(81^{n-1} + 81^{n-2} \cdot 8 + \dots + 8^{n-1})$$

(and it is 0 for $n = 0$), both $64(81^n - 8^n)$ and $73 \cdot 8^n$ are divisible by 73, and so is N .

8. The functions $f(x) = \sin x$ and $g(x) = \cos x$ are defined on the interval $[0, 2\pi]$.

a) Find the equations of the tangent lines to the graph of f that are parallel to the line $x - 2y = 0$.

b) What is the area of the region bounded by the graphs of f and g ?

(16 points)

Solution: a) The slope of the line $x - 2y = 0$ is $\frac{1}{2}$, and the slope of the tangent at x is $f'(x) = \cos x$. So $\cos x = \frac{1}{2}$, and in $[0, 2\pi]$, $x = \frac{\pi}{3}$ or $x = \frac{5\pi}{3}$. The points of tangency are $\left(\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$ and $\left(\frac{5\pi}{3}, -\frac{\sqrt{3}}{2}\right)$, and the tangent lines are

$$x - 2y = \frac{\pi}{3} - \sqrt{3} \quad \text{and} \quad x - 2y = \frac{5\pi}{3} + \sqrt{3}$$

b) In $[0, 2\pi]$ the two graphs meet at $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$, and between these points $\sin x \geq \cos x$. So the area is

$$T = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x) dx = [-\cos x - \sin x]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = 2\sqrt{2}$$

9. The 12 lattice points (points with both coordinates integers) on the circle of radius 5 centered at the origin determine a twelve-sided polygon.

a) Show that this polygon has no inscribed circle.

b) Find the area of the polygon.

c) A margarine box in the shape of a frustum of a pyramid, 6 cm high, is designed with a lid similar to this polygon, of area 74 cm^2 . What will the area of the base of the box be, if it is designed to hold 4 deciliters?

(16 points)

Solution: The lattice points are $(\pm 5, 0)$, $(0, \pm 5)$, $(\pm 4, \pm 3)$, $(\pm 3, \pm 4)$.

a) Three neighboring vertices are $A(5, 0)$, $B(4, 3)$, $C(3, 4)$, with $AB = \sqrt{10}$ and $BC = \sqrt{2}$. The polygon is symmetric about the origin, so if it had an inscribed circle, its center would be the origin, and the distances of the origin from the lines AB and BC would be equal. But by the Pythagorean Theorem these distances (the altitudes of the isosceles triangles OAB and OBC) are

$$OT_1 = \sqrt{25 - \frac{10}{4}} = \frac{3}{2}\sqrt{10}, \quad OT_2 = \sqrt{25 - \frac{2}{4}} = \frac{7}{2}\sqrt{2}$$

which are different. So the polygon has no inscribed circle.

b) The polygon is made up of 8 triangles congruent to OAB and 4 triangles congruent to OBC , with areas

$$t_{OAB} = \frac{\sqrt{10} \cdot \frac{3}{2}\sqrt{10}}{2} = \frac{15}{2}, \quad t_{OBC} = \frac{\sqrt{2} \cdot \frac{7}{2}\sqrt{2}}{2} = \frac{7}{2}$$

so its area is $T = 8 \cdot \frac{15}{2} + 4 \cdot \frac{7}{2} = 74$ square units.

c) The volume of the box is 400 cm^3 . By the volume formula of the frustum, $V = \frac{m}{3} (T + \sqrt{Tt} + t)$, where t is the area of the base:

$$400 = \frac{6}{3} (74 + \sqrt{74t} + t), \quad t + \sqrt{74t} - 126 = 0$$

This is a quadratic equation in $\sqrt{t}$, and its positive root gives $\sqrt{t} \approx 7.72$, so the area of the base is $t \approx 59.6 \text{ cm}^2$.