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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. Find the possible values of the positive integers a and b , if $a < b$, and whenever x and y are elements of the interval $[a, b]$, then $\frac{1}{x} + \frac{1}{y}$ is also an element of $[a, b]$.

(12 points)

Solution: $a > 1$ is impossible: then $a \geq 2$, so $x \geq 2$ and $y \geq 2$, and $\frac{1}{x} + \frac{1}{y} \leq 1 < a$. So $a = 1$.

$b > a$ gives $b \geq 2$, and $b > 2$ is impossible: then $x = y = 2.5$ are in $[a, b]$, but $\frac{1}{2.5} + \frac{1}{2.5} = 0.8 < 1 = a$. So $b = 2$.

$a = 1, b = 2$ is indeed good: if $1 \leq x \leq 2$ and $1 \leq y \leq 2$, then $1 \leq \frac{1}{x} + \frac{1}{y} \leq 2$. So $a = 1$ and $b = 2$.

2. a) Solve the equation

$$\sqrt{1 + x\sqrt{x^2 - 16}} = x - 1$$

over the real numbers.

(5 points)

- b) Find the domain of the above equation.

(7 points)

Solution: a) If x is a solution, then squaring both sides, $1 + x\sqrt{x^2 - 16} = x^2 - 2x + 1$, that is, $x\sqrt{x^2 - 16} = x^2 - 2x$. $x = 0$ is not a solution of the original equation, so $\sqrt{x^2 - 16} = x - 2$, and squaring again, $x^2 - 16 = x^2 - 4x + 4$, so $x = 5$. Substitution shows that $x = 5$ is indeed a solution ($\sqrt{1 + 15} = 4$).

b) The domain consists of the real numbers x with $x^2 - 16 \geq 0$, that is, $|x| \geq 4$, and $x\sqrt{x^2 - 16} \geq -1$. The latter always holds for $x \geq 4$; for $x \leq -4$ it holds exactly when $|x|\sqrt{x^2 - 16} \leq 1$, that is, $x^4 - 16x^2 - 1 \leq 0$, or $x^2 \leq 8 + \sqrt{65}$. So the domain is

$$\left[-\sqrt{8 + \sqrt{65}}, -4\right] \cup [4, \infty)$$

3. A car dealer sells five kinds of cars. It sells one car of each kind for

2.3, 2.8, 3, 3.1, 3.5

million Ft, and its profit (the difference of the selling price and the purchase price) is, for the kinds in the order above,

3, 2.5, 2.4, 2.1, 2

percent of the selling price. These two lists of data do not change during the investigation. Of each kind (in the order above) it sold

in 2003 25 32 27 38 19 cars,
in 2004 27 31 29 32 33 cars.

- a) What was its profit in 2003, and in 2004?

(5 points)

- b) If in 2005 it sells only one kind of car, which kind (with what unit price) is it worth selling, and at least how many of them must it sell, if it wants to reach the profit of 2004 by selling as few cars as possible?

(7 points)

Solution: a) The profit in 2003 (in millions of Ft) was

$$2.3 \cdot 0.03 \cdot 25 + 2.8 \cdot 0.025 \cdot 32 + 3 \cdot 0.024 \cdot 27 + 3.1 \cdot 0.021 \cdot 38 + 3.5 \cdot 0.02 \cdot 19 = 9.7128$$

and in 2004 it was

$$2.3 \cdot 0.03 \cdot 27 + 2.8 \cdot 0.025 \cdot 31 + 3 \cdot 0.024 \cdot 29 + 3.1 \cdot 0.021 \cdot 32 + 3.5 \cdot 0.02 \cdot 33 = 10.5142$$

- b) The profit on one car is 0.069, 0.07, 0.072, 0.0651, and 0.07 million Ft for the five kinds, so the largest is 0.072, for the 3 million Ft car. It must sell at least as many of these as needed for the profit to reach the profit of 2004: $\frac{10.5142}{0.072} = 146.03 \dots$, so 147 cars.

4. The circles of radii r and R are externally tangent to each other, and $r < R$. The angle of their common external tangents is α .

- a) What is $\frac{R}{r}$, if $\alpha = 60^\circ$?

(5 points)

- b) What is α , if $\frac{R}{r} = \frac{5}{3}$?

(5 points)

- c) What is the area of the quadrilateral two of whose vertices are the centers of the circles, and the other two are the points of tangency on one of the common external tangents, if $R = 5$ and $r = 3$?

(5 points)

Solution: Let the centers be O_1 (small circle) and O_2 , and the points of tangency on one common external tangent A and B . The line through O_1 parallel to AB meets O_2B at B_1 ; the right triangle $O_1B_1O_2$ has hypotenuse $O_1O_2 = R + r$ and leg $O_2B_1 = R - r$, and its angle at O_1 is $\frac{\alpha}{2}$ (since O_1O_2 bisects the angle of the tangents). So

$$\frac{R - r}{R + r} = \sin \frac{\alpha}{2}$$

a) $\sin 30^\circ = \frac{1}{2}$, so $2(R - r) = R + r$, and $\frac{R}{r} = 3$.

b) $\sin \frac{\alpha}{2} = \frac{\frac{R}{r} - 1}{\frac{R}{r} + 1} = \frac{1}{4}$, so $\alpha \approx 28.96^\circ$.

c) The quadrilateral is a right trapezoid with parallel sides R and r , and the leg perpendicular to them is $AB = O_1B_1 = \sqrt{(R + r)^2 - (R - r)^2} = 2\sqrt{Rr}$. So its area is

$$\frac{5 + 3}{2} \cdot 2\sqrt{15} = 8\sqrt{15} \approx 30.98$$

Part II

5. The set H is a subset of the set of positive integers not greater than 2005 such that the sum of any two of its elements is not divisible by 3. At most how many elements does H have?

(14 points)

Solution: Consider the remainders of the numbers when divided by 3. At most one number with remainder 0 can be in H , since the sum of two such numbers is divisible by 3. Numbers with remainders 1 and 2 cannot both occur, since such a sum is divisible by 3; but any number of numbers with the same remainder 1 (or 2) can be in H , together with one number with remainder 0. From 1 to 2004 there are as many numbers with remainder 1 as with remainder 2, namely 668, and 2005 has remainder 1. So the maximum is reached by choosing the 669 numbers with remainder 1 and one number with remainder 0: H has at most 670 elements.

6. There are five red balls in a box.

a) How many white balls must we put into the box, if we want that drawing one ball at random from the box (each ball equally likely), the ball drawn is red with probability 0.25?

(6 points)

b) At least how many white and at least how many black balls must we put into the box (at least one of each), if we want that drawing one ball at random, the ball drawn is not black with probability 0.25?

(10 points)

Solution: a) With x white balls, the probability of red is $\frac{5}{5 + x} = 0.25$, so $x = 15$.

b) With x white and y black balls, the probability that the ball is not black is $\frac{5 + x}{5 + x + y} = 0.25$, that is, $y = 3x + 15$. The smallest numbers with $x \geq 1$ are $x = 1$ and $y = 18$.

7. Solve the inequality

$$\log_x (x^2 + x - 4) < 1$$

over the real numbers.

(16 points)

Solution: Because of the base, only $x > 0$, $x \neq 1$ can occur, and we need $x^2 + x - 4 > 0$, that is, $x < \frac{-1 - \sqrt{17}}{2}$ or $x > \frac{-1 + \sqrt{17}}{2}$. With $x > 0$, this means $x > \frac{\sqrt{17} - 1}{2} \approx 1.56$. Then the base is greater than 1, and the logarithm is less than 1 exactly when $x^2 + x - 4 < x$, that is, $-2 < x < 2$. The solution is

$$\frac{\sqrt{17} - 1}{2} < x < 2$$

8. Let A be the right triangular prism whose base is an equilateral triangle with side 2 and whose lateral edges are also 2 long. Rotate the solid A by 90° around the line through the center of one of its lateral faces, perpendicular to that face, and let A' be the rotated solid. What is

a) the volume of the solid $A \cup A'$?

(8 points)

b) the surface area of the solid $A \cup A'$?

(8 points)

Solution: $A \cap A'$ is a square pyramid whose base is the common lateral face (a 2×2 square) and whose height is the height of the base triangle of the prism, $\sqrt{3}$.

a) The volume of $A \cap A'$ is $\frac{2 \cdot 2 \cdot \sqrt{3}}{3} = \frac{4\sqrt{3}}{3} \approx 2.31$, so the volume of $A \cup A'$ is

$$2 \cdot \sqrt{3} \cdot 2 - \frac{4\sqrt{3}}{3} = \frac{8\sqrt{3}}{3} \approx 4.62$$

(the volume of the prism is the area of the base, $\sqrt{3}$, times the height 2). (We checked this value numerically.)

b) The common lateral face (4 square units) is on the surface of both solids. Of each slanted lateral face of A , the half that is inside A' (area 2) disappears, and the other half (area 2) remains; the two triangular bases of A are entirely outside A' . The same holds for A' . So the surface area is $4 + 2(2 + 2 + 2\sqrt{3}) = 12 + 4\sqrt{3} \approx 18.93$.

9. a) Write the equations of the parabolas whose graph is tangent to the line $y = 2 - 2x$ at the point $P(1, 0)$, and whose axis is parallel to the y -axis.

(9 points)

b) Where are the vertices of these parabolas?

(9 points)

Solution: a) Let the parabola be $y = ax^2 + bx + c$ ($a \neq 0$). It passes through $P(1, 0)$, so $c = -a - b$. The line and the parabola have only one common point, so the equation $2 - 2x = ax^2 + bx - a - b$ has only one root: its discriminant is

$$(b + 2)^2 + 4a(a + b + 2) = (b + 2 + 2a)^2 = 0$$

so $b = -2a - 2$. The parabolas are

$$y = ax^2 - (2a + 2)x + a + 2, \quad a \neq 0$$

b) The vertex of such a parabola is at $x = \frac{a+1}{a}$, and its y -coordinate is $a\left(\frac{a+1}{a}\right)^2 - (2a+2)\frac{a+1}{a} + a + 2 = -\frac{1}{a}$.

So the vertices are the points $\left(1 + \frac{1}{a}, -\frac{1}{a}\right)$, which lie on the line $x + y = 1$; every point of this line except P (where $\frac{1}{a} = 0$) is the vertex of one of the parabolas.