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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. Find the four consecutive odd numbers the sum of whose squares is 48 more than the sum of the squares of the even numbers between them.

Solution: Let the four consecutive odd numbers be $a - 3$, $a - 1$, $a + 1$, $a + 3$. Then

$$(a - 3)^2 + (a - 1)^2 + (a + 1)^2 + (a + 3)^2 = (a - 2)^2 + a^2 + (a + 2)^2 + 48$$

that is, $4a^2 + 20 = 3a^2 + 8 + 48$, so $a^2 = 36$, $a = 6$ or $a = -6$. The numbers are 3, 5, 7, 9, or -9 , -7 , -5 , -3 .

2. On the sides AB , BC , CD , and DA of the unit square $ABCD$ we take the points E , F , G , H so that $AE = \frac{1}{2}$, $BF = \frac{1}{3}$, $CG = \frac{2}{3}$, and $DH = \frac{1}{2}$. Find the angles, the perimeter, and the area of the quadrilateral $EFGH$.

Solution: $EFGH$ is an isosceles trapezoid ($EH \parallel FG$), with angles 78.69° and 101.31° . By the Pythagorean Theorem its sides are

$$EH = \frac{\sqrt{2}}{2}, \quad FG = \frac{2}{3}\sqrt{2}, \quad EF = HG = \frac{\sqrt{13}}{6}$$

The perimeter is $k = \frac{7\sqrt{2} + 2\sqrt{13}}{6} \approx 2.85$, and the area is what remains of the square after cutting off the four corner triangles:

$$t = 1 - \frac{1}{8} - 2 \cdot \frac{1}{12} - \frac{2}{9} = \frac{35}{72}$$

3. We form six-digit numbers from the digits 1, 2, 3.
 - a) How many different numbers can we get?
 - b) How many of them contain each of the three digits at least once?
 - c) What is the probability that the number we get is even?
 - d) What is the probability that the number we get is divisible by 3?

Solution: a) $3^6 = 729$.

b) The bad numbers are those made of a single digit (3 of them) and those containing exactly two different digits ($3(2^6 - 2)$ of them). So the number we want is $3^6 - 3 - 3(2^6 - 2) = 540$.

c) The number is even if its last digit is 2; there are 3^5 such numbers, so the probability is $\frac{3^5}{3^6} = \frac{1}{3}$.

d) For any choice of the first five digits, exactly one choice of the last digit makes the number divisible by 3 (if the sum of the first five digits gives remainder 0, 1, or 2 when divided by 3, the last digit must be 3, 2, or 1). So 3^5 numbers are divisible by 3, and the probability is $\frac{1}{3}$.

4. Solve the following equation over the real numbers.

$$\log_{\sin x} (1 - \cos^2 x) = 2^{\sqrt{4-x^2}}$$

Solution: The equation is defined if $\sin x > 0$, $\sin x \neq 1$, $1 - \cos^2 x = \sin^2 x > 0$, and $4 - x^2 \geq 0$; these hold if $x \in (0, 2]$ and $x \neq \frac{\pi}{2}$. Then the left-hand side is $\log_{\sin x} \sin^2 x = 2$, so the equation is $2 = 2^{\sqrt{4-x^2}}$, that is, $\sqrt{4-x^2} = 1$, $x = \pm\sqrt{3}$. Because of the domain, the solution is $x = \sqrt{3}$.

Part II

5. Péter's library of 1000 volumes consists of books in Hungarian, English, and German. $p\%$ of the books are in Hungarian, $p\%$ of the foreign-language books are in English, and he has only 10 books in German. Find the number of Hungarian and English books.

Solution: Let a be the number of English and m the number of Hungarian books. Then $a + m = 990$, $m = 1000 \cdot \frac{p}{100}$, and $a = (a + 10) \frac{p}{100}$. Rearranging, we get $m^2 - 2000m + 990\,000 = 0$, whose solutions are 1100 and 900. Only 900 makes sense, so there are 900 Hungarian and 90 English books ($p = 90$).

6. Let

$$f(x) = \frac{1 - \frac{4x}{x+1}}{\frac{x}{x+1} - \frac{1}{1-x} - \frac{2x}{x^2-1}}$$

- What is the domain of the expression above?
- Bring the expression to the simplest possible form.
- Through how many lattice points does the graph of the function $f(x)$ pass?

Solution: a) The fractions are defined for $x \neq \pm 1$. The denominator of f is

$$\frac{x(x-1) + (x+1) - 2x}{(x+1)(x-1)} = \frac{(x-1)^2}{(x+1)(x-1)} = \frac{x-1}{x+1}$$

which is not 0 there, so the domain is $\mathbb{R} \setminus \{-1, 1\}$.

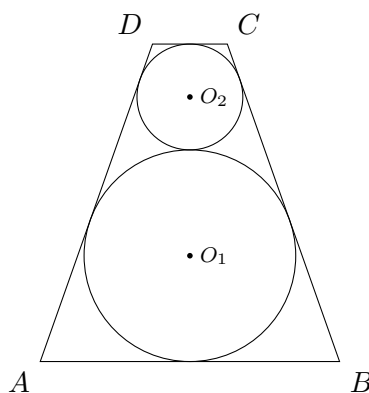
- b) The numerator is $\frac{1-3x}{x+1}$, so $f(x) = \frac{1-3x}{x-1}$.

c) $f(x) = \frac{-3(x-1)-2}{x-1} = -3 - \frac{2}{x-1}$. Its graph is a hyperbola, and it passes through a lattice point if x is an integer and $x-1$ is a divisor of 2, that is, $x-1 = -2, -1, 1, 2$, or $x = -1, 0, 2, 3$. But $x = -1$ is not in the domain, so the graph passes through three lattice points: $(0, -1)$, $(2, -5)$, $(3, -4)$.

(Note: the KoMaL solution also lists $(-1, -2)$, but $x = -1$ is excluded from the domain.)

7. A statue consists of two balls placed on top of each other, where the radius of the upper ball is half the radius of the lower ball, and the height of the statue is 3 meters. To protect the statue against the hardships of winter, a competition was announced for making a “protective cover”. Two entries arrived: one proposed a cover in the shape of a frustum of a square pyramid, the other in the shape of a frustum of a cone. In both, the lateral surface touches both balls, and the top touches the smaller ball (there is no bottom base). The cost of the cone frustum design is 12 000 Ft/m², and that of the pyramid frustum is 10 000 Ft/m². Which design would be cheaper to make?

Solution: Consider the plane section through the centers of the balls (for the pyramid, the plane that bisects two opposite sides of the base and the top). In both cases this is the same isosceles trapezoid $ABCD$, in which the two circles of radii $2r$ and r touch each other, the legs, and the circle of radius r also touches CD :



Let r be the radius of the smaller ball, O_1 and O_2 the centers, F and E the midpoints of AB and CD , and let the leg AD touch the circles at points that are x from A and y from D , with d the distance of the two points of tangency on AD : $d = \sqrt{9r^2 - r^2} = 2\sqrt{2}r$. Then $AF = x$, $DE = y$ (tangent segments), and $AD = x + d + y$. From similar right triangles, $\frac{2r}{y} = \frac{x}{r}$, that is, $xy = 2r^2$; and by the Pythagorean Theorem (the height of the trapezoid is $6r$),

$(x + d + y)^2 = (x - y)^2 + (6r)^2$. So $x = 2\sqrt{2}r$, $y = \frac{\sqrt{2}}{2}r$. The height of the statue is $2r + 4r = 3$, so $r = 0.5$ m.

The cone frustum has bottom radius x , top radius y , and slant height $x + y + d$; its surface (top and lateral surface) is

$$A_{\text{cone}} = \pi y^2 + \pi (x + y) (x + y + d) = \frac{23}{4}\pi \approx 18.06 \text{ m}^2$$

The pyramid frustum has bottom edge $2x$, top edge $2y$, and the height of its lateral faces is $x + y + d$:

$$A_{\text{pyramid}} = (2y)^2 + 4 \cdot \frac{(2x + 2y)(x + y + d)}{2} = 23 \text{ m}^2$$

The cone frustum design costs about $18.06 \cdot 12\,000 \approx 216\,800$ Ft, and the pyramid frustum design $23 \cdot 10\,000 = 230\,000$ Ft, so the cone frustum design is cheaper.

8. Two trucks arrive at a logistics center at random times within a 24-hour period. Unloading starts as soon as a truck arrives; it takes 1 hour for one of them and 2 hours for the other. If the second truck arrives while the other is still being unloaded, it has to wait until the unloading is finished. What is the probability that one of the trucks has to wait for unloading?

Solution: Let x be the arrival time of the truck that can be unloaded in 1 hour and y that of the other, $0 \leq x \leq 24$, $0 \leq y \leq 24$; we represent the arrivals by the points (x, y) of a square. If $x < y$, one truck has to wait if $y < x + 1$; if $x > y$, if $x < y + 2$. These points form a strip along the diagonal, and the two triangles outside it have legs 23 and 22. So the probability is

$$\frac{24 \cdot 24 - \frac{1}{2} \cdot 23 \cdot 23 - \frac{1}{2} \cdot 22 \cdot 22}{24 \cdot 24} = \frac{69.5}{576} \approx 0.12$$

9. The n th term of a sequence is $a_n = \frac{2}{(2n+1)(2n+3)}$, where n is a positive integer.

a) Find b and c so that $a_n = \frac{b}{2n+1} - \frac{c}{2n+3}$ for every n .

b) Find the fifth decimal digit of the sum of the first 2005 terms.

Solution: a) $\frac{b}{2n+1} - \frac{c}{2n+3} = \frac{2n(b-c) + 3b-c}{(2n+1)(2n+3)}$, and this equals a_n for every n if $b=c$ and $3b-c=2$, that is, $b=c=1$.

b) The sum telescopes:

$$S_{2005} = \frac{1}{3} - \frac{1}{5} + \frac{1}{5} - \frac{1}{7} + \dots + \frac{1}{4011} - \frac{1}{4013} = \frac{1}{3} - \frac{1}{4013} = \frac{4010}{12039} = 0.333084143\dots$$

so the fifth decimal digit is 8.