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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the standard-level exam takes 180 minutes. Part I (45 minutes, 30 points) consists of short problems; only the final answer has to be given, except where the problem asks for a justification. Part II (135 minutes) consists of Part II/A, three problems worth 36 points in all, and Part II/B, three problems of 17 points each, of which students choose and solve 2, for 34 points. So there is a total of 100 points possible. Calculators and four-digit function tables may be used.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present the problems.

Part I

1. In the store we can buy 4 kilograms of bread for 500 forints. How many kilograms of bread can we buy for the same money, if the price of bread is raised by 20%? (2 points)

Solution: 1 kg of bread cost 125 Ft, and the new price is 150 Ft, so 500 Ft buys $\frac{500}{150} = 3\frac{1}{3}$ kg of bread.

2. Find the largest subset of the real numbers on which the expression $\frac{1}{\sqrt{x^2 - 4}}$ is defined. (2 points)

Solution: Because of the square root and the denominator, $x^2 - 4 > 0$, that is, $x > 2$ or $x < -2$: $(-\infty, -2) \cup (2, \infty)$.

3. We put cards with the numbers 1, 2, 3, 4, 5, 6 into an urn, and draw two of them without replacement. Writing the drawn numbers next to each other in the order we choose, we form a two-digit number. What is the probability that we get a number divisible by 6 this way? (3 points)

Solution: (We take the order of the digits to be the order of drawing.) Taking the order into account, there are $6 \cdot 5 = 30$ equally likely outcomes. The two-digit numbers with different digits from 1–6 that are divisible by 6 are 12, 24, 36, 42, 54, so the probability is $\frac{5}{30} = \frac{1}{6}$.

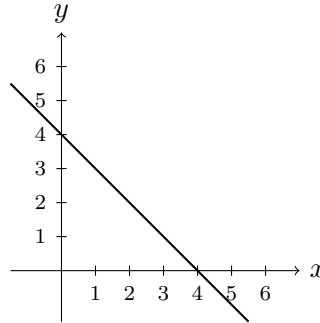
(Note: the KoMaL solution counts 10 good outcomes and gets $\frac{1}{3}$; this is a slip: for example, 21, 63, and 45 are not divisible by 6.)

4. The points $A(3, -6)$ and $B(7, -3)$ are given in the coordinate system. Find the length of the segment AB . (2 points)

Solution: $AB = \sqrt{(7-3)^2 + (-3+6)^2} = 5$.

5. Give the rule of the linear function shown in the figure.

(2 points)



Solution: $x \mapsto 4 - x$.

6. The following seven numbers are given: 3, 2, 3, 7, 1, 9, 10. Find the mean and the median of the data.

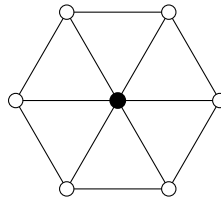
(2 points)

Solution: The mean is $\frac{35}{7} = 5$; in increasing order the data are 1, 2, 3, 3, 7, 9, 10, so the median is 3.

7. Smallcountry has seven settlements in all. The capital is connected to every settlement by a road, and apart from the capital, from every settlement one can get directly to two other settlements by road. Draw a simple graph that represents the road network of Smallcountry.

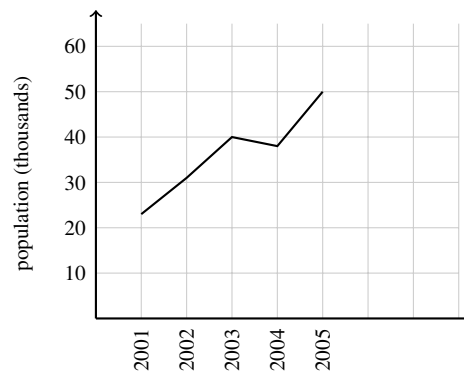
(2 points)

Solution: The capital is joined to the other six settlements, and the six other settlements form a cycle (each is joined to two others):



(The capital is the vertex in the middle. The six other settlements could also form two triangles.)

8. The graph shows the population of a town in the years indicated. Determine, for the period shown,



- a) between which two years the increase in population was the largest, and approximately how large this increase was;

(2 points)

- b) between which two years the population decreased.

(1 points)

Solution: a) Between 2004 and 2005; the population grew by about 12 000.

- b) Between 2003 and 2004.

9. “Every spotted magpie has a spotted tail.” Which of the following statements means the same as this statement? (3 points)

- A) There is a spotted magpie whose tail is spotted.
- B) Every magpie with a spotted tail is spotted.
- C) There is no spotted magpie whose tail is not spotted.
- D) Every magpie has a spotted tail.
- E) For every magpie it is true that if its tail is spotted, then it is spotted itself.

(The original Hungarian sentence, “Minden tarka szarka farka tarka”, is a well-known tongue twister.)

Solution: If the original statement is true, then there cannot be a spotted magpie whose tail is not spotted; and if there is no spotted magpie whose tail is not spotted, then every spotted magpie has a spotted tail. So the correct answer is C).

10. a) Simplify the following algebraic fraction.

(2 points)

$$\frac{a^2 - b^2}{a + b} \cdot \frac{a + b}{5a - 5b}$$

- b) Find the value of the expression for $a = 150$, $b = 55$.

(1 points)

Solution: a) $\frac{(a+b)(a-b)}{a+b} \cdot \frac{a+b}{5(a-b)} = \frac{a+b}{5}$ (for $a \neq \pm b$).

b) $\frac{205}{5} = 41$.

11. One leg of a right triangle is 5 cm, and the other leg is 12 cm long.

- a) What is the area of the triangle?

(1 points)

- b) What is the altitude to the hypotenuse? Justify your answer.

(2 points)

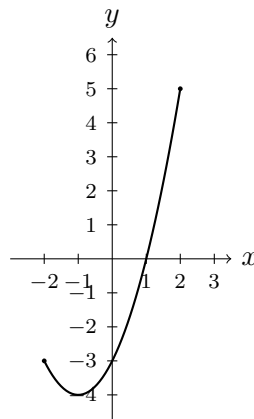
Solution: a) The area is half the product of the legs: 30 cm^2 .

- b) By the Pythagorean Theorem the hypotenuse is 13 cm, so the altitude to it is $m = \frac{2T}{13} = \frac{60}{13} \approx 4.62 \text{ cm}$.

12. Graph the function $x \mapsto x^2 + 2x - 3$ on the interval $[-2, 2]$.

(3 points)

Solution: $x^2 + 2x - 3 = (x + 1)^2 - 4$: a parabola with vertex $(-1, -4)$, from $(-2, -3)$ to $(2, 5)$, with zero at $x = 1$.



Part II/A

13. A company rents out photocopiers. There are two types: for the Mismásoló the monthly rent is 5000 Ft and 8 Ft must be paid per copy; for the Másmásoló the monthly rent is 9500 Ft and 5 Ft must be paid per copy. (There are no other costs.)

- a) By what percent is copying more or less expensive with the Mismásoló than with the Másmásoló, for 1000 copies a month? (2 points)
- b) As a function of the monthly number of copies, which copier is it worth choosing? (4 points)
- c) We want to open a copy shop, and we expect to make 20 000 copies a month there. What price should we set for one copy, if we want a 20% profit compared to our copying costs? (We rent the Másmásoló for the shop.) (3 points)
- d) In our copy shop we signed a flat-rate rental contract for the machine: we prepay 20 000 copies a month according to the rental terms of the Másmásoló, and therefore we copy as many as we want for this amount. We decide to sell a copy for 10 forints. In one month we have already sold 10 000 copies, and the last day of the month has come. We receive an order for 5000 copies. For how much should we take one copy, so that we make a 20% profit on the money invested in this month? (3 points)

Solution: a) For 1000 copies a month, the Mismásoló costs $5000 + 1000 \cdot 8 = 13\,000$ Ft and the Másmásoló $9500 + 1000 \cdot 5 = 14\,500$ Ft. Since $\frac{13\,000}{14\,500} \approx 0.897$, copying with the Mismásoló is about 10.3% cheaper.

b) The difference of 4500 Ft in the basic rent is made up by the Másmásoló at 1500 copies. Above 1500 copies a month the Másmásoló, below 1500 copies the Mismásoló is better (at exactly 1500 copies they cost the same).

c) 20 000 copies cost $9500 + 20\,000 \cdot 5 = 109\,500$ Ft with the Másmásoló. With a 20% profit the total is 131 400 Ft, which is 6.57 Ft per copy.

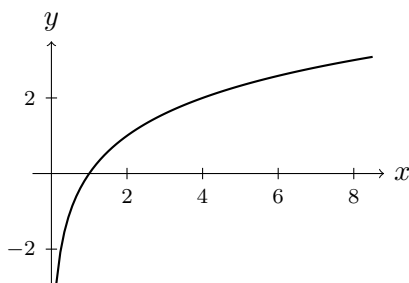
d) We paid 109 500 Ft for the 20 000 copies, and we want an income of 131 400 Ft. So far we have taken in 100 000 Ft, so the 5000 copies must bring the remaining 31 400 Ft: we should charge 6.28 Ft per copy.

14. a) Graph the function $x \mapsto \log_2 x$ defined on the positive real numbers. (4 points)
- b) Solve the equation

$$4^{x+\frac{3}{2}} - 33 \cdot 2^x + 2^2 = 0$$

over the real numbers. (8 points)

Solution: a) The graph passes through $\left(\frac{1}{4}, -2\right)$, $\left(\frac{1}{2}, -1\right)$, $(1, 0)$, $(2, 1)$, $(4, 2)$, $(8, 3)$; it is increasing, and the y -axis is an asymptote.



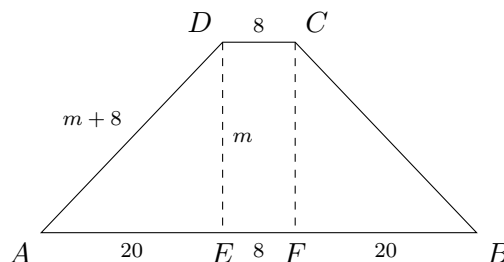
- b) Since $4^{x+\frac{3}{2}} = 8 \cdot 4^x$, with the new variable $a = 2^x$ the equation is $8a^2 - 33a + 4 = 0$, whose roots are 4 and $\frac{1}{8}$.
From $2^x = 4$ and $2^x = \frac{1}{8}$, the solutions are $x = 2$ and $x = -3$.

15. The parallel sides of an isosceles trapezoid are 8 and 48 units long. Its legs are equal to the sum of the shorter parallel side and the height.

a) Find the area of the trapezoid. (6 points)

b) Find the radius of the circle circumscribed around the trapezoid. (6 points)

Solution: a) Let m be the height of the trapezoid $ABCD$, and E, F the projections of D and C on AB ; then $AE = 20$ and the leg is $AD = m + 8$.



From the right triangle DAE : $(m + 8)^2 = m^2 + 400$, so $m = 21$. The area of the trapezoid is $\frac{8 + 48}{2} \cdot 21 = 588$.

b) The center O of the circumscribed circle is on the axis of symmetry. Let G and H be the midpoints of CD and AB , and r the radius. O is below AB (the angles at C and D are obtuse); with $OH = x$, from the right triangles CGO and BHO :

$$r^2 = (x + 21)^2 + 4^2, \quad r^2 = x^2 + 24^2$$

So $x = \frac{119}{42}$, and $r = \frac{145}{6} \approx 24.17$.

Part II/B

Choose and solve two of the problems 16, 17, and 18.

16. Solve the following equations over the real numbers.

a) $\sin 2x = \frac{1}{2}$ (6 points)

b) $\log_2(x + 1) + \frac{1}{2} \log_2(x^2 - 8x + 16) = 1 + \log_2 3$ (11 points)

Solution: a) $2x = \frac{\pi}{6} + 2k\pi$ or $2x = \frac{5\pi}{6} + 2k\pi$, so $x = \frac{\pi}{12} + k\pi$ or $x = \frac{5\pi}{12} + k\pi$, where k is any integer.

b) The equation is defined for $x > -1$, $x \neq 4$. With equivalent steps: $2 \log_2(x + 1) + \log_2(x - 4)^2 = 2 \log_2 6$, that is, $[(x + 1)(x - 4)]^2 = 36$, since $\log_2$ is one-to-one. This holds exactly when $(x + 1)(x - 4) = 6$ or $(x + 1)(x - 4) = -6$. The first gives $x = 5$ and $x = -2$ (not a solution, because of the domain); the second gives $x = 1$ and $x = 2$, both solutions. The solutions are 1, 2, and 5.

(The equation can also be written as $\log_2(x + 1) + \log_2|x - 4| = \log_2 6$.)

17. After a tanker accident, three ships want to localize an oil slick that got into the sea, with the help of a rope covered with sponge stretched between them. The pairwise distances between the ships are 40, 60, and 80 meters. The oil slick fills the part between the sponge ropes.

a) What is the area of the oil slick? (7 points)

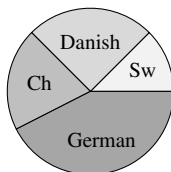
b) Of the 20 crew members of one of the ships, 15 refuse to work, citing bad food. The captain randomly assigns 2 sailors to the pump. What is the probability that exactly one of those assigned is a striker? (4 points)

c) Of the 80 crew members of the largest ship, 10 are Swedish, 20 Danish, 16 Chinese, and 34 German. Show the composition of the crew in a pie chart. (6 points)

Solution: a) In the triangle formed by the rope, by the Law of Cosines for the angle opposite the 80 m side: $40^2 + 60^2 - 2 \cdot 40 \cdot 60 \cos \alpha = 80^2$, so $\cos \alpha = -0.25$ and $\alpha \approx 104.48^\circ$. The area of the triangle (the oil slick) is $\frac{1}{2} \cdot 40 \cdot 60 \cdot \sin \alpha \approx 1161.9 \text{ m}^2$.

b) Two sailors can be chosen in $\binom{20}{2} = 190$ ways; one striker and one non-striker in $15 \cdot 5 = 75$ ways. The probability is $\frac{75}{190} = \frac{15}{38} \approx 0.39$.

c) The central angles are: Swedish 45° , Danish 90° , Chinese 72° , German 153° .



18. Two dogs are tied in the coordinate plane: one to the point $(4, 5)$ with a leash of length 2, one end of which can rotate freely around the point; the other to the line $3x + 4y = 7$ with a leash of length 1, one end of which can slide freely along the line. What is the smallest possible distance between the two dogs? (17 points)

Solution: The first dog moves within the closed disk of radius 2 around $(4, 5)$, and the second within the strip of points at most 1 from the line. The distance of $(4, 5)$ from the line is $\frac{|12 + 20 - 7|}{5} = 5$. (The foot of the perpendicular from $(4, 5)$ to the line is $(1, 1)$, on the line $4x - 3y = 1$.) The smallest distance occurs when both dogs are on this perpendicular, and it is $5 - 2 - 1 = 2$ units.