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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the standard-level exam takes 180 minutes. Part I (45 minutes, 30 points) consists of short problems; only the final answer has to be given, except where the problem asks for a justification. Part II (135 minutes) consists of Part II/A, three problems worth 36 points in all, and Part II/B, three problems of 17 points each, of which students choose and solve 2, for 34 points. So there is a total of 100 points possible. Calculators and four-digit function tables may be used.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 180 minutes to solve and present the problems.

Part I

1. How many of the following expressions are equal to the sum $x^x + x^x$ for every real number $x > 0$?

I. $2 \cdot x^x$ II. x^{2x} III. $(2x)^x$ IV. $(2x)^{2x}$

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

(1 point)

Solution: The answer is (B). I. $x^x + x^x = 2x^x$. II. $x^{2x} = (x^x)^2 \neq 2x^x$ in general. III. $(2x)^x = 2^x x^x \neq 2x^x$. IV. $(2x)^{2x} = 2^{2x} x^{2x} \neq 2x^x$.

2. Find the value of the following expression. Justify your answer.

$$\left(\frac{3}{5}\right)^{2005} : \frac{3^{2006} + 15^{2006}}{5^{2006} + 5^{4012}}$$

(A) 1 (B) 5 (C) $\frac{3}{5}$ (D) $\frac{5}{3}$ (E) 3

(2 points)

Solution: The answer is (D):

$$\left(\frac{3}{5}\right)^{2005} \cdot \frac{5^{2006} + 5^{4012}}{3^{2006} + 15^{2006}} = \frac{3^{2005} \cdot 5^{2006} (1 + 5^{2006})}{5^{2005} \cdot 3^{2006} (1 + 5^{2006})} = \frac{5}{3}$$

3. Anna and Béla started working at a chain of stores on March 1, 2005. Anna's schedule is that she gets a day off after three working days, while Béla gets three days off after seven working days. In the following 120 days, how many times do their days off fall on the same day? (2 points)

Solution: Anna works in a 4-day and Béla in a 10-day cycle, so we compute the least common multiple of 4 and 10: the schedules repeat every 20 days. Within one 20-day cycle 2 days off are common (the 8th and the 20th day). In 120 days there are 6 such cycles, so their days off coincide 12 times.

4. The coordinates of the four vertices of the quadrilateral $ABCD$ are $A(-3, 1)$, $B(-4, -5.5)$, $C(8, -1)$, $D(5, 4)$. Which of the following statements is true?

a) The quadrilateral $ABCD$ is a parallelogram. b) The quadrilateral $ABCD$ is a trapezoid. (2 points)

Solution: $\overrightarrow{AD} = (8, 3)$ and $\overrightarrow{BC} = (12, 4.5)$, so $\overrightarrow{BC} = \frac{3}{2}\overrightarrow{AD}$: $AD \parallel BC$ but $AD \neq BC$. So the quadrilateral is a trapezoid and not a parallelogram: b) is true.

5. In 2004 a bicycle cost 32 000 Ft and a bicycle helmet 8000 Ft. In 2005 the price of the bicycle increased by 5%, and the price of the helmet by 10%. By what percent did the combined price of the bicycle and the helmet increase in 2005?
- (A) 6% (B) 7% (C) 7.5% (D) 8% (E) 15% (2 points)

Solution: The answer is (A): $\frac{32\,000 \cdot 1.05 + 8000 \cdot 1.1}{40\,000} = \frac{42\,400}{40\,000} = 1.06$, so the increase was 6%.

6. The finite sets A , B , and C are given:

$$A = \{x \mid x \in \mathbb{N}, x^2 \leq 4\}, \quad B = \{x \mid x \in \mathbb{Z}, |x| = 2\}, \quad C = \{x \mid x \text{ is a prime and } 0 < x < 6\}$$

- a) List the elements of the sets A , B , and C .
 b) Give the elements of the set $A \cap B \cap C$.
 c) Is it true that B is a subset of A ? Justify your statement. (3 points)

Solution: a) $A = \{0, 1, 2\}$ (in Hungary 0 is a natural number), $B = \{-2, 2\}$, $C = \{2, 3, 5\}$.

b) $A \cap B \cap C = \{2\}$.

c) No: $-2 \in B$ but $-2 \notin A$.

7. For which real numbers are the following expressions defined? a) $\sqrt{x^2 - 1}$ b) $\log_{\frac{1}{3}} \sqrt{1 - x}$ (2 points)

Solution: a) $x^2 - 1 \geq 0$, that is, $x \geq 1$ or $x \leq -1$. b) $1 - x > 0$, that is, $x < 1$.

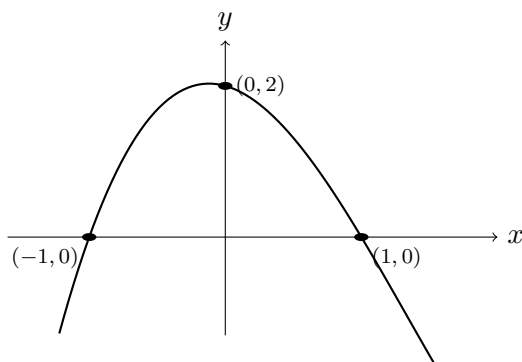
8. Solve the equation $4^x - 2^{x+1} = 3$ over the real numbers. (3 points)

Solution: The equation is $2^{2x} - 2 \cdot 2^x - 3 = 0$. With $a = 2^x$ we get $a^2 - 2a - 3 = 0$, whose roots are 3 and -1 . Since $2^x > 0$, only $2^x = 3$ gives a solution: $x = \log_2 3 = \frac{\lg 3}{\lg 2} \approx 1.585$.

9. Anna caught a cold, got a fever and a sore throat. The doctor prescribed two kinds of capsules: a green one and a white one. She has to take 3 green and 2 white capsules a day for two weeks. She paid 1470 Ft for the two kinds of medicine at the pharmacy. The pharmacist told her that one white capsule costs 10 Ft less than one green capsule. How much does one green and one white capsule cost? (3 points)

Solution: The medicine for one day costs $\frac{1470}{14} = 105$ Ft. If a green capsule costs x Ft, then a white one costs $x - 10$ Ft, and $3x + 2(x - 10) = 105$, so $x = 25$. A green capsule costs 25 Ft and a white one 15 Ft.

10. The figure shows a part of the graph of the cubic function $f(x) = ax^3 + bx^2 + cx + d$. What is the value of b ?



- (A)
- -4
- (B)
- -2
- (C)
- 0
- (D)
- 1
- (E)
- 4

(3 points)

Solution: The answer is (B). From the figure, $f(0) = 2$, so $d = 2$; $f(-1) = 0$, so $0 = -a + b - c + 2$; $f(1) = 0$, so $0 = a + b + c + 2$. Adding the last two, $2b + 4 = 0$, so $b = -2$.

11. An ice cream seller pressed a spherical scoop of strawberry ice cream into a cone-shaped wafer (a right circular cone). The diameter of the cone and of the ice cream ball are equal. When the ice cream ball melts, it exactly fills the cone. The volume of the melted ice cream is 75% of the volume of the frozen ice cream. Find the ratio of the height of the cone to the radius of the ice cream ball. (3 points)

Solution: Let r be the common radius of the cone and the ball, and m the height of the cone. Then

$$\frac{3}{4} \cdot \frac{4r^3\pi}{3} = \frac{r^2\pi m}{3}, \quad m = 3r$$

So the ratio of the height of the cone to the radius of the ball is 3 : 1.

12. $ABCD$ is a convex quadrilateral. We extend each side of the quadrilateral, in the same direction around the quadrilateral, by its own length. What is the ratio of the area of the resulting quadrilateral $A'B'C'D'$ to the area of $ABCD$? Justify your answer.

- (A) 4 : 1 (B) 5 : 1 (C) 6 : 1 (D) 7 : 1 (E) 8 : 1

(4 points)

(The number of points is missing in KoMaL; 4 points make Part I add up to 30.)

Solution: The answer is (B). Let A' be the point with B the midpoint of AA' , and so on (B' on the extension of BC , C' of CD , D' of DA). The quadrilateral $A'B'C'D'$ consists of $ABCD$ and four triangles, for example $A'BB'$. Since a median divides a triangle into two triangles of equal area, the area of $A'BB'$ is twice the area of ABC (it is twice the area of $A'BC$, which equals the area of ABC). Similarly the four triangles have areas $2T_{ABC}$, $2T_{BCD}$, $2T_{CDA}$, $2T_{DAB}$, and $T_{ABC} + T_{CDA} = T_{BCD} + T_{DAB} = T_{ABCD}$. So

$$T_{A'B'C'D'} = T_{ABCD} + 2 \cdot 2T_{ABCD} = 5T_{ABCD}$$

and the ratio is 5 : 1.

Part II/A

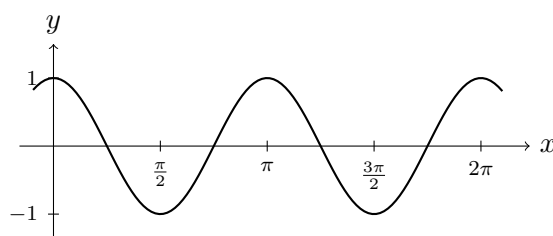
13. a) Graph the function $x \mapsto \cos 2x$ defined on the real numbers, and find the places, the types, and the values of the extrema of the function $x \mapsto \cos 2x$. (6 points)

b) Solve the following equation over the real numbers:

$$\cot^2 x - 8 \sin^2 x = 1$$

(6 points)

Solution: a) We obtain the graph of $x \mapsto \cos 2x$ from the graph of $x \mapsto \cos x$ by shrinking it to half in the direction of the x -axis:



The function has (local and global) maxima at $x = k\pi$ with value 1, and minima at $x = \frac{\pi}{2} + k\pi$ with value -1 (k an integer).

b) Using $\cot x = \frac{\cos x}{\sin x}$ ($\sin x \neq 0$) and multiplying by $\sin^2 x$: $\cos^2 x - 8\sin^4 x = \sin^2 x$, that is, $8\sin^4 x + 2\sin^2 x - 1 = 0$. So $\sin^2 x = \frac{-2 \pm 6}{16}$: $\sin^2 x = \frac{1}{4}$, or $\sin^2 x = -\frac{1}{2}$, which is impossible. So $\sin x = \pm \frac{1}{2}$, and the solutions are $x = \pm \frac{\pi}{6} + k\pi$ (k an integer). All of them satisfy the original equation.

14. The lines $e: x - 2y = 3$ and $f: 3x - y = 4$ meet at the point M .

a) Find the coordinates of the intersection point M .

b) Write the equation of the line h that passes through M and is perpendicular to the line $g: 2x - 3y = 1$.

c) Give the coordinates of the intersection points of the line h with the coordinate axes. (12 points)

Solution: a) From $x - 2y = 3$, $x = 2y + 3$, and substituting into $3x - y = 4$: $y = -1$, $x = 1$. So $M(1, -1)$.

b) A normal vector of g is $(2, -3)$, so a normal vector of h is $(3, 2)$, and h passes through $M(1, -1)$: $3x + 2y = 3 - 2 = 1$, or $y = -\frac{3}{2}x + \frac{1}{2}$.

c) h meets the y -axis at $\left(0, \frac{1}{2}\right)$ and the x -axis at $\left(\frac{1}{3}, 0\right)$.

15. In the triangle ABC the side AB is 1 cm and the side AC is 2 cm long, and the side BC is equal to the length of the median from the vertex A . How long is the side BC of the triangle ABC ? (12 points)

Solution: Let $BC = 2a$, let F be the midpoint of BC , and $\delta = \angle BFA$; then $\angle AFC = 180^\circ - \delta$ and $AF = 2a$. By the Law of Cosines in the triangles ABF and AFC :

$$a^2 + 4a^2 - 4a^2 \cos \delta = 1, \quad a^2 + 4a^2 + 4a^2 \cos \delta = 4$$

Adding them, $10a^2 = 5$, so $a = \frac{\sqrt{2}}{2}$, and $BC = 2a = \sqrt{2} \approx 1.4$ cm.

Part II/B

Choose and solve two of the problems 16, 17, and 18.

16. We write the numbers 1, 2, 3 on three slips of paper, one on each, and put the three slips into a hat. We draw a slip from the hat, record the number on it, and put the slip back into the hat. We repeat the drawing twice more. Each slip is equally likely to be drawn. If the sum of the recorded numbers is 6, what is the probability that the slip with the number 2 was drawn three times? (17 points)

Solution: The outcomes with sum 6 are

$$(1, 2, 3), (1, 3, 2), (2, 3, 1), (2, 1, 3), (3, 1, 2), (3, 2, 1), (2, 2, 2)$$

7 equally likely outcomes, of which one is favorable, $(2, 2, 2)$. The probability is $\frac{1}{7}$.

17. The terms $a_1, a_2, a_3, \dots, a_n, \dots$ of an arithmetic sequence satisfy

$$a_4 + a_7 + a_{10} = 17, \quad a_4 + a_5 + a_6 + \dots + a_{13} + a_{14} = 77$$

a) Find the seventh term of the sequence. b) Find the ninth term of the sequence. c) Give the difference of the sequence.

d) Find the first term of the sequence. e) Determine which term of the sequence is 13. (17 points)

Solution: a) a_4, a_7, a_{10} are three consecutive terms of an arithmetic sequence (with difference $3d$), so $3a_7 = 17$, and $a_7 = \frac{17}{3}$.

b) $a_4, \dots, a_{14}$ are 11 terms with middle term a_9 , so $11a_9 = 77$, and $a_9 = 7$.

c) $a_9 - a_7 = 2d$, so $d = \frac{2}{3}$.

d) $a_7 = a_1 + 6d$, so $a_1 = \frac{17}{3} - 4 = \frac{5}{3}$.

e) $a_k = a_1 + (k-1)d$: $13 = \frac{5}{3} + (k-1) \cdot \frac{2}{3}$, so $k = 18$: $a_{18} = 13$.

18. In a pheasant farm in the Pallag nature reserve, the number of pheasants grows so that the difference between the population in year $n+2$ and in year n is directly proportional to the population in year $n+1$.

a) If there were 39 pheasants in 2001, 60 in 2002, and 123 in 2004, then how many were there in 2003?

b) Show in a bar chart how the yearly number of pheasants changed between 2001 and 2004 in the Pallag nature reserve. (17 points)

Solution: a) Let x be the number in 2003, and k the factor of proportionality. Then $x - 39 = 60k$ and $123 - 60 = kx$. So $k = \frac{63}{x}$, $x - 39 = 60 \cdot \frac{63}{x}$, that is, $x^2 - 39x - 3780 = 0$, whose roots are 84 and -45 ; only 84 makes sense. There were 84 pheasants in 2003.

b) The bar chart:

