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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. a) $A = 101^{100^{99}}$, $B = 99^{100^{101}}$. Which number is larger, A or B ?

(7 points)

- b) Write the equation of the tangent line to the graph of $f(x) = x^2$ that makes a 60° angle with the positive half of the x -axis.

(6 points)

Solution: a) The base 10 logarithm is strictly increasing, so it is enough to compare the logarithms $\lg A = 100^{99} \cdot \lg 101$ and $\lg B = 100^{101} \cdot \lg 99$. These are positive, so we compare their logarithms again:

$$\lg(\lg A) = 99 \cdot \lg 100 + \lg(\lg 101) = 198 + \lg(\lg 101) < 198 + \lg 10 = 199$$

$$\lg(\lg B) = 101 \cdot \lg 100 + \lg(\lg 99) = 202 + \lg(\lg 99) > 202$$

(since $1 < \lg 99 < \lg 101 < 10$). So $\lg A < \lg B$, and $A < B$.

- b) The slope of the tangent is $\tan 60^\circ = \sqrt{3}$, and $f'(x) = 2x$, so the point of tangency has $x = \frac{\sqrt{3}}{2}$ and $y = \frac{3}{4}$. The tangent line is

$$y = \sqrt{3} \left(x - \frac{\sqrt{3}}{2} \right) + \frac{3}{4} = \sqrt{3}x - \frac{3}{4}$$

2. Where is the mistake in the following arguments? What is the solution of the equation in a), and what is the probability in question in b)?

- a) The equation $\sqrt{x} = -x$ has no solution, because the square root of a number cannot be negative.

(4 points)

- b) What is the probability that, rolling seven fair dice, there is a 1 among the numbers? The probability of rolling a 1 with a single die is $\frac{1}{6}$. So the probability that rolling two dice there is a 1 among the numbers is twice as much, $\frac{1}{3}$; so the probability in question is 7 times as much, $\frac{7}{6}$: the event is more than certain.

(5 points)

- c) One meter is 100 centimeters, so $\frac{1}{4}$ meter = 25 centimeters. The square root of $\frac{1}{4}$ is $\frac{1}{2}$, the square root of 25 is 5, so $\frac{1}{2}$ meter = 5 centimeters.

(4 points)

Solution: a) $-x$ is not negative if $x \leq 0$, so the argument is wrong. The equation requires $x \geq 0$ and $-x \geq 0$, so the solution is $x = 0$ (and indeed $\sqrt{0} = 0 = -0$).

b) The probability of success is not proportional to the number of trials. The probability that there is no 1 among seven dice is $\left(\frac{5}{6}\right)^7$, so the probability in question is $1 - \left(\frac{5}{6}\right)^7 \approx 0.7209$.

c) The measure of a quantity is the product of a number and a unit. Taking the square roots of the numbers gives a true equality only if the unit is the same on both sides. (In fact $\sqrt{\frac{1}{4} \text{ m}^2} = \frac{1}{2} \text{ m}$, and $\sqrt{25 \text{ cm}^2} = 5 \text{ cm}$, but $\frac{1}{4} \text{ m}^2 \neq 25 \text{ cm}^2$.)

3. The height, one diagonal, and the other diagonal of a trapezoid are, in this order, three consecutive terms of a geometric sequence with ratio $q = 2$. The area of the trapezoid is $T = \sqrt{60} + \sqrt{12}$ square units. What is the height of the trapezoid?

(12 points)

Solution: Let the trapezoid be $ABCD$ ($AB \parallel CD$) with height m ; by the conditions $AC = 2m$ and $BD = 4m$. Let C_1 and D_1 be the projections of C and D on the line AB . By the Pythagorean Theorem $AC_1 = \sqrt{3}m$ and $BD_1 = \sqrt{15}m$. Since $C_1D_1 = CD$, the sum of the parallel sides is

$$AB + CD = AC_1 + BD_1 = \sqrt{3}m + \sqrt{15}m = \sqrt{3}m(1 + \sqrt{5})$$

Twice the area is $\sqrt{3}m(1 + \sqrt{5}) \cdot m = 2(\sqrt{60} + \sqrt{12}) = 4\sqrt{3}(\sqrt{5} + 1)$, so $m^2 = 4$, and the height is $m = 2$.

4. Solve the following inequalities over the real numbers.

a) $2x^4 \leq x^2 + 1$

(5 points)

b) $2 \cos^2 x \geq \cos x + 1$

(8 points)

Solution: a) With $t = x^2 \geq 0$ the inequality is $2t^2 - t - 1 \leq 0$, whose solution is $-\frac{1}{2} \leq t \leq 1$. Since $t \geq 0$, we need $0 \leq x^2 \leq 1$, that is, $-1 \leq x \leq 1$.

b) With $t = \cos x$ ($-1 \leq t \leq 1$) we get $2t^2 - t - 1 \geq 0$, whose solution is $t \leq -\frac{1}{2}$ or $t \geq 1$. If $\cos x \leq -\frac{1}{2}$, then $\frac{2\pi}{3} + 2k\pi \leq x \leq \frac{4\pi}{3} + 2k\pi$; if $\cos x \geq 1$, then $\cos x = 1$, so $x = 2n\pi$ (k, n integers).

Part II

5. a) The sum of the positive integers A and B is 1000. Prove that the last three digits of A^2 are the same as the last three digits of B^2 .

(4 points)

b) Give positive integers A and B with sum 1000 such that the last *four* digits of A^2 are the same as the last *four* digits of B^2 .

(3 points)

c) How many positive divisors does the number $H = 2^{24} - 1$ have?

(9 points)

Solution: a) The statement holds if $1000 \mid A^2 - B^2$, and indeed $A^2 - B^2 = (A + B)(A - B) = 1000(A - B)$.

b) The last four digits agree if $10\,000 \mid A^2 - B^2 = 1000(A - B)$, that is, if $A - B$ is divisible by 10. With $A + B = 1000$, this holds for example if both A and B are divisible by 10, for example $A = 110$, $B = 890$.

c) Factoring:

$$H = (2^{12} - 1)(2^{12} + 1) = (2^3 - 1)(2^3 + 1)(2^2 + 1)(2^4 - 2^2 + 1)(2^4 + 1)(2^8 - 2^4 + 1)$$

that is, $H = 7 \cdot 9 \cdot 5 \cdot 13 \cdot 17 \cdot 241 = 3^2 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 241$. The number of positive divisors is $(2 + 1) \cdot 2^5 = 96$.

6. A box contains coins and balls, made of silver or gold. 20% of the objects in the box are balls, 40% of the coins are silver, and 80% of the gold objects are coins.

a) What percent of the objects in the box are gold coins?

(4 points)

b) We draw an object from the box at random. What is the probability that it is made of silver?

(4 points)

c) We draw a ball from the box at random. What is the probability that it is made of silver?

(4 points)

d) Interpret the results of b) and c) for the following events: E : the object drawn is made of silver; G : the object drawn is a ball.

(4 points)

Solution: Let t be the number of objects, and arrange the data in a table, using that the row and column sums add up to the number of objects:

	ball	coin	total
gold	$0.12t$	$0.48t$	$0.6t$
silver	$0.08t$	$0.4 \cdot 0.8t = 0.32t$	$0.4t$
total	$0.2t$	$0.8t$	t

(The coins are $0.8t$, of which $0.32t$ are silver and $0.48t$ gold; the gold coins are 80% of the gold objects, so there are $0.6t$ gold objects.)

a) 48%. b) 0.4. c) $\frac{0.08t}{0.2t} = 0.4$.

d) The proportion of silver objects among the balls is the same as among all objects in the box: the events E and G are independent, that is, $P(E \mid G) = P(E)$, or $P(EG) = P(E)P(G)$.

7. The first term of an arithmetic sequence is 9, and its ninth term is 33.

a) Find the sum of those terms of the sequence that are between 149 and 301.

(10 points)

b) What is the sum of the squares of the first twenty-five terms of the sequence?

(6 points)

Solution: a) $d = \frac{33 - 9}{8} = 3$, so $a_n = 9 + (n - 1) \cdot 3 = 3n + 6$. The n th term is between 149 and 301 if $149 < 3n + 6 < 301$, that is, $\frac{143}{3} < n < \frac{295}{3}$, or $47 < n < 99$. So 51 terms are between the given bounds, from $a_{48} = 150$ to $a_{98} = 300$, and their sum is

$$51 \cdot \frac{150 + 300}{2} = 11\,475$$

b) $a_1^2 + \dots + a_{25}^2 = \sum_{i=1}^{25} (3i + 6)^2 = 9 \sum_{i=1}^{25} (i + 2)^2$, and $\sum_{i=1}^{25} (i + 2)^2 = 3^2 + \dots + 27^2$ is $1^2 + 2^2 = 5$ less than the sum of the first 27 squares, $\frac{27 \cdot 28 \cdot 55}{6} = 6930$. So the sum is $9 \cdot 6925 = 62\,325$.

8. In a clinic, a blood test is done to detect a disease that, on average, one person in a hundred has. The people to be tested arrive in groups of fifty, and the practice so far has been to test each of them individually. Testing one blood sample costs 300 forints.

The head of the clinic received two proposals for changing this:

- A) Set aside part of each blood sample of a group of 50 in an identifiable way, mix the other parts, and test the group sample first. If it is negative, then obviously everyone is healthy; if positive, test the individual samples set aside.
- B) Similarly to A), first test the combined sample of the group of 50. If it is positive, then make two group samples of 25 people each from the individual samples set aside, and test these. If either group turns out to be positive, then test the members of that group individually.

As the head of the clinic, would you change the practice? If yes, how and why?

(16 points)

Solution: Each person is sick with probability 0.01, independently of the others. In the original procedure, testing a group of 50 costs $50 \cdot 300 = 15\,000$ Ft. We compare the expected costs of A) and B).

- A) The probability that nobody in a group of 50 is sick is $p = 0.99^{50} \approx 0.605$. The expected cost is

$$E_A = p \cdot 300 + (1 - p) \cdot 51 \cdot 300 \approx 6225 \text{ Ft}$$

- B) The probability that nobody in a group of 25 is sick is $p_1 = 0.99^{25} \approx 0.778$. The number of tests needed: if nobody in the group of 50 is sick (probability $p_1^2 = p \approx 0.605$), 1 test; if exactly one half is sick (probability $2p_1(1 - p_1) \approx 0.345$), $1 + 2 + 25 = 28$ tests; if both halves are sick (probability $(1 - p_1)^2 \approx 0.05$), $1 + 2 + 50 = 53$ tests. The expected cost is

$$E_B = 300(1 \cdot 0.605 + 28 \cdot 0.345 + 53 \cdot 0.05) \approx 3870 \text{ Ft}$$

Both are much cheaper than 15 000 Ft, and B) is the cheapest, so it is worth choosing B).

(Note: the KoMaL solution counts 2, 27, and 52 tests in the three cases, that is, it leaves out the test of the group of 50 and always tests the two groups of 25, and gets $E_B \approx 3937.5$ Ft. The conclusion is the same.)

9. Calculators may *not* be used for the following problems. Justify your results.

- a) 1. Write the sum of the first four terms of the expansion of $(1 + x)^{10}$, in increasing powers of x .

(2 points)

2. Using this expression, give an approximation of 1.005^{10} .

(1 points)

3. Prove that the approximation obtained this way is accurate to four decimal places.

(5 points)

- b) 1. Sketch the graph of the function $f(x) = \frac{1500x + 1}{500x - 2}$ for $x > 1000$.

(2 points)

2. Find the integer part of the positive solution of the equation

$$\frac{1500x + 1}{500x - 2} = \frac{3x}{10\,000}$$

(6 points)

Solution: a) 1. $1 + 10x + \binom{10}{2}x^2 + \binom{10}{3}x^3 = 1 + 10x + 45x^2 + 120x^3$.

2. $1.005^{10} \approx 1 + 10 \cdot 0.005 + 45 \cdot 0.000\,025 + 120 \cdot 0.000\,000\,125 = 1.05114$.

3. The remaining 7 terms are $\binom{10}{k} 0.005^k$ for $k = 4, \dots, 10$; the largest binomial coefficient is $\binom{10}{5} = 252$, and the powers of 0.005 decrease, so each of these terms is at most $252 \cdot 0.005^4$, and their sum is less than

$$7 \cdot 252 \cdot (5 \cdot 10^{-3})^4 < 10^{-8} \cdot 1764 < \frac{10^{-4}}{2}$$

So the sum of the first four terms is accurate to four decimal places.

b) 1. $f(x) = 3 + \frac{7}{500x - 2}$, so for $x > 1000$ the graph is indistinguishable from its horizontal asymptote $y = 3$, and it is a little above it (since $f(x) > 3$ for $x > \frac{1}{250}$).

2. If instead of the given equation we solve $3 = \frac{3x}{10\,000}$, we get 10 000; since the left-hand side of the original equation is greater than 3 and the right-hand side is increasing, the real solution is greater than 10 000. On the other hand, substituting $x = 10\,001$, the right-hand side 3.0003 is greater than the left-hand side $3 + \frac{7}{5\,000\,498}$, so the positive root is less than 10 001. Its integer part is 10 000.