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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. A cyclist covered $\frac{1}{4}$ of the distance between the towns A and B in 1 hour, and the rest of the way in 4 hours. His speed (in km/h) was an integer on both parts, and the least common multiple of the two speeds is 72. How long is the distance AB ?

(11 points)

Solution: On the first part the cyclist covered $\frac{1}{4}$ of the distance in 1 hour, and on the second part $\frac{3}{4}$ of it in 4 hours, so the ratio of the speeds is $\frac{v_1}{v_2} = \frac{1/4}{3/16} = \frac{4}{3}$. So $v_1 = 4k$ and $v_2 = 3k$ for a positive integer k (4 and 3 are relatively prime), and $[4k, 3k] = 12k = 72$, so $k = 6$. So $v_1 = 24$ km/h, $v_2 = 18$ km/h, and the distance is $1 \cdot 24 + 4 \cdot 18 = 96$ km.

2. a) Solve the following system of equations over the pairs of real numbers.

$$\frac{1}{1 + \tan x} + \frac{1}{1 + \cot x} = xy, \quad xy + 4 = 2(x + y)$$

(7 points)

- b) The first term of an arithmetic sequence is the larger of the solutions x, y of the system, and its difference is the smaller one. How many two-digit positive integer terms does the sequence have?

(6 points)

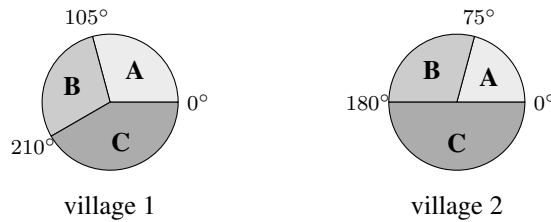
Solution: a) Where both sides are defined,

$$\frac{1}{1 + \tan x} + \frac{1}{1 + \frac{1}{\tan x}} = \frac{1}{1 + \tan x} + \frac{\tan x}{\tan x + 1} = 1$$

so $xy = 1$, and the second equation gives $5 = 2\left(x + \frac{1}{x}\right)$, that is, $2x^2 - 5x + 2 = 0$. The solutions are $x_1 = 2, y_1 = \frac{1}{2}$ and $x_2 = \frac{1}{2}, y_2 = 2$. (For these values $\tan x$ and $\cot x$ are defined and different from -1 .)

- b) In both cases the first term is 2 and the difference is $\frac{1}{2}$, so $a_n = 2 + \frac{n-1}{2}$. This is an integer exactly when n is odd, and every integer from 2 on occurs. So the two-digit positive integer terms are 10, 11, ..., 99: 90 terms.

3. Three villages elect a common mayor. There are three candidates (**A**, **B**, and **C**). In village 1, 2016 voters, and in village 2, 1320 voters went to the polls. The pie charts show the results in villages 1 and 2 (the angles mark the boundaries of the sectors). Unfortunately the pie chart showing the result in village 3 was lost.



The combined final result of the election: **A**: 1599 votes, **B**: 2169 votes, **C**: 1776 votes.

- a) Make the pie chart showing the result of the election in village 3.

(10 points)

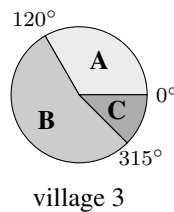
- b) How many inhabitants does village 3 have, if 72% of its inhabitants were eligible to vote, and 56% of the eligible voters took part in the election?

(4 points)

Solution: a) From the pie charts, the sectors in village 1 are 105°, 105°, 150°, and in village 2 they are 75°, 105°, 180°. So the votes were

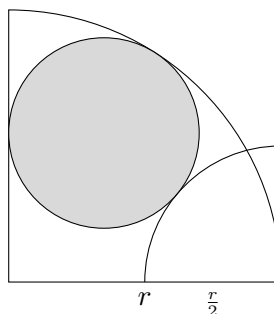
village 1: **A**: 588, **B**: 588, **C**: 840; village 2: **A**: 275, **B**: 385, **C**: 660.

Subtracting these from the final result, village 3 gave **A**: 736, **B**: 1196, **C**: 276 votes, 2208 in all. The angles of the sectors are $\frac{736}{2208} \cdot 360^\circ = 120^\circ$, 195° , and 45° :



- b) If x is the number of inhabitants, then $x \cdot 0.72 \cdot 0.56 = 2208$, so $x \approx 5476.2$. Village 3 has about 5476 inhabitants.

4. The figure shows one half of a decorative element of a chapel to be built in a small village (the element consists of two mirror-image copies of this part, on the two sides of a cross). The large quarter circle has radius $r = 80$ cm; a small quarter circle of radius $\frac{r}{2}$ is centered at the other end of its horizontal radius, and the shaded circle (a glass window) touches the vertical radius, the large arc, and the small arc. What is the radius of the shaded circles?



(14 points)

Solution: Let O be the center of the large quarter circle, Q the center of the small one ($OQ = r$), and let x be the radius of the glass window and (x, y) its center (O is the origin). The window touches the large arc from inside and the small arc from outside, so its center is $r - x$ from O and $\frac{r}{2} + x$ from Q . By the Pythagorean Theorem

$$x^2 + y^2 = (r - x)^2 \quad \text{and} \quad (r - x)^2 + y^2 = \left(\frac{r}{2} + x\right)^2$$

Substituting $y^2 = (r - x)^2 - x^2$ into the second equation: $2(r - x)^2 - x^2 = \left(\frac{r}{2} + x\right)^2$, which simplifies to $\frac{7}{4}r^2 = 5rx$. So $x = \frac{7}{20}r$, and the radius of the glass windows is $\frac{7 \cdot 80}{20} = 28$ cm.

Part II

5. Let A be the domain of the expression (1), and B the domain of the expression (2):

$$(1) \quad \sqrt{2 \sin^2 \pi x - \cos \pi x - 1}, \quad (2) \quad \lg(-3x^2 + 17x - 10)$$

(lg is the base 10 logarithm). Find

a) the elements of the set $A \cap B$;

(11 points)

b) the elements of the set $B \setminus A$.

(5 points)

Solution: a) A is the solution set of $2 \sin^2 \pi x - \cos \pi x - 1 \geq 0$, that is, $-2 \cos^2 \pi x - \cos \pi x + 1 \geq 0$, or $(2 \cos \pi x - 1)(\cos \pi x + 1) \leq 0$. This means $-1 \leq \cos \pi x \leq \frac{1}{2}$, that is, $\cos \pi x \leq \frac{1}{2}$:

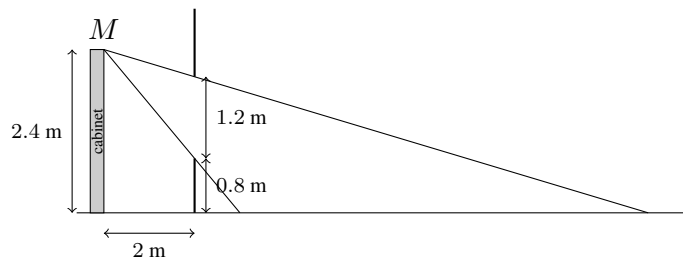
$$\frac{\pi}{3} + 2k\pi \leq \pi x \leq \frac{5\pi}{3} + 2k\pi, \quad \text{so} \quad \frac{1}{3} + 2k \leq x \leq \frac{5}{3} + 2k \quad (k \in \mathbb{Z})$$

B is the solution set of $-3x^2 + 17x - 10 > 0$; the zeros are $\frac{2}{3}$ and 5, so $B = \left\{x : \frac{2}{3} < x < 5\right\}$. So

$$A \cap B = \left\{x : \frac{2}{3} < x \leq \frac{5}{3}\right\} \cup \left\{x : \frac{7}{3} \leq x \leq \frac{11}{3}\right\} \cup \left\{x : \frac{13}{3} \leq x < 5\right\}$$

$$b) \quad B \setminus A = \left\{x : \frac{5}{3} < x < \frac{7}{3}\right\} \cup \left\{x : \frac{11}{3} < x < \frac{13}{3}\right\}.$$

6. A bird sits at the corner M of a 2.4 m high cabinet that stands 2 m from a wall, and through a 1.2 m high window in the wall it watches a bug that crawls in the yard, perpendicular to the wall of the house, at a speed of 0.8 m/min. The lower edge of the window is 0.8 m above the ground. For how long does the bird see the bug?



(16 points)

Solution: The bird sees the bug while it is between the points where the lines from M through the lower and upper edges of the window reach the ground. Let these points be y and $x + y$ meters from the wall. By similar triangles (the cabinet is 2 m from the wall):

$$\frac{y}{0.8} = \frac{y+2}{2.4}, \quad y = 1, \quad \frac{x+y}{0.8+1.2} = \frac{x+y+2}{2.4}, \quad x+y = 10$$

so $x = 9$ m. The bird sees the bug for $t = \frac{9 \text{ m}}{0.8 \text{ m/min}} = 11.25$ minutes.

7. For the functions $f(x) = x^2 + ax + b$ and $g(x) = x^2 + bx + a$ ($a > b$) there is exactly one point x_0 at which the tangent lines to the two graphs are perpendicular.

a) Sketch the graph of the function $h(x) = f(x) - g(x)$.

(8 points)

b) Find the area of the region bounded by the graphs of f and g and the y -axis.

(8 points)

Solution: a) The condition is that $f'(x_0) \cdot g'(x_0) = -1$ has exactly one solution: $(2x_0 + a)(2x_0 + b) = -1$, that is, $4x_0^2 + 2(a+b)x_0 + ab + 1 = 0$. This has exactly one solution if its discriminant is 0: $4(a+b)^2 - 16(ab+1) = 0$, that is, $(a-b)^2 = 4$. Since $a > b$, $a = b + 2$, and

$$h(x) = (a-b)x + b - a = 2x - 2$$

The graph of h is the line through $(1, 0)$ and $(0, -2)$.

b) The graphs of f and g meet at $x = 1$, and for $x < 1$, $g(x) > f(x)$. So the area is

$$\int_0^1 (g(x) - f(x)) dx = \int_0^1 (2 - 2x) dx = [2x - x^2]_0^1 = 1$$

(it does not depend on b).

8. András saw a chestnut-cream bunny at a very good price in an advertising paper. When he wanted to buy it, he was shocked to find that at the cash register he was charged 66 Ft less than twice the price in the paper. After he complained, it turned out that the first and third digits of the three-digit price had been switched by mistake in the paper. How much does András have to pay, if he buys the item?

(16 points)

Solution: If András saw the price $\overline{abc}$ in the paper, then the real price is $\overline{cba} = 2 \cdot \overline{abc} - 66$, that is, $\overline{cba} + 66 = 2 \cdot \overline{abc}$. The right-hand side is even, so a is even (the last digit of the left-hand side is that of $a + 6$). Also $a < 6$, otherwise the right-hand side would be greater than 1200. So $a = 2$ or $a = 4$.

If $a = 4$, the left-hand side ends in 0, so $c = 5$; but then the right-hand side is greater than 800 and the left-hand side is less than 700, which is impossible.

If $a = 2$, the left-hand side ends in 8, so $c = 4$ or $c = 9$; $c = 9$ is impossible, since then the left-hand side is greater than 900 and the right-hand side is less than 600. So $c = 4$, and $4b2 + 66 = 2 \cdot 2b4$, that is, $468 + 10b = 408 + 20b$, and $b = 6$.

The price in the paper was 264 Ft, and the real price is 462 Ft. Indeed, $2 \cdot 264 - 66 = 462$.

9. a) Three parallel lines are given; on each of them we painted 5 points red. Consider all triangles with red vertices that have two vertices on one of the lines and the third vertex on another line; then consider all quadrilaterals with red vertices that have two vertices on each of two of the lines. Which are there more of: triangles or quadrilaterals?

(6 points)

- b) Now let two parallel lines be given, with n red points on one ($n \geq 2$) and $n + 1$ red points on the other. Again we form all triangles and all quadrilaterals with red vertices. Now which are there more of, triangles or quadrilaterals?

(10 points)

Solution: a) The number of triangles is $3 \cdot \binom{5}{2} \cdot 10 = 300$ (the line with two vertices, the two vertices, and the third vertex on one of the other two lines), and the number of quadrilaterals is $3 \cdot \binom{5}{2}^2 = 300$ (the two lines and two points on each). So there are as many triangles as quadrilaterals.

- b) The number of triangles is $\binom{n}{2}(n+1) + \binom{n+1}{2}n$, and the number of quadrilaterals is $\binom{n}{2}\binom{n+1}{2}$. There are more triangles if

$$\frac{(n-1)n(n+1)}{2} + \frac{n^2(n+1)}{2} > \frac{(n-1)n^2(n+1)}{4}, \quad \frac{n-1}{2} + \frac{n}{2} > \frac{n(n-1)}{4}, \quad n^2 - 5n + 2 < 0$$

The zeros of $n^2 - 5n + 2$ are $\frac{5 \pm \sqrt{17}}{2} \approx 0.44$ and 4.56 . So for $n = 2, 3, 4$ there are more triangles, and for $n \geq 5$ there are more quadrilaterals. (For example, for $n = 4$ there are 70 triangles and 60 quadrilaterals, and for $n = 5$ there are 135 triangles and 150 quadrilaterals.)