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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. In a right triangle the ratio of the legs is $1 : 2$, and the perimeter and the area have the same measure. Find the exact length of the altitude to the hypotenuse.

(11 points)

Solution: Let the legs be a and $2a$; then the hypotenuse is $a\sqrt{5}$, the area is a^2 , and the perimeter is $a(3 + \sqrt{5})$. From $a^2 = a(3 + \sqrt{5})$ and $a \neq 0$, $a = 3 + \sqrt{5}$. The area is also $\frac{a\sqrt{5} \cdot m}{2}$, where m is the altitude to the hypotenuse, so

$$\frac{(3 + \sqrt{5})\sqrt{5} \cdot m}{2} = (3 + \sqrt{5})^2, \quad m = \frac{2(3 + \sqrt{5})}{\sqrt{5}} = \frac{6\sqrt{5} + 10}{5}$$

2. Solve the following system of equations.

$$x^2 - y^2 - 6x - 6y = 0, \quad xy - x + y = 0$$

(12 points)

Solution: The first equation can be written as $(x + y)(x - y - 6) = 0$, so $y = -x$ or $y = x - 6$. Substituting into the second equation:

If $y = -x$: $x^2 + 2x = 0$, so $(x, y) = (0, 0)$ or $(-2, 2)$.

If $y = x - 6$: $x^2 - 6x - 6 = 0$, so $(x, y) = (3 + \sqrt{15}, -3 + \sqrt{15})$ or $(3 - \sqrt{15}, -3 - \sqrt{15})$.

3. In a single round-robin volleyball cup (where any two teams play each other exactly once), after 30 matches had been played, every team still had three matches left. How many teams took part in the cup?

(14 points)

Solution: Let the number of teams be n . The number of all matches is $\frac{n(n-1)}{2}$. Every team had 3 matches left, which is $\frac{3n}{2}$ matches in all. So

$$\frac{n(n-1)}{2} - \frac{3n}{2} = 30, \quad n^2 - 4n - 60 = 0$$

whose roots are 10 and -6 . 10 teams took part in the cup.

4. The production of a factory grew in five consecutive years: by 12% in the first year, by 15% in the second, and by 20% in the third. In the fourth and fifth years the percentage of growth was the same. In the fifth year the factory reached 2.3 times the production of the year before the period examined. By what percent did production grow in the fourth and fifth years?

(14 points)

Solution: Let the production in the year before the period be 1, and let $q = 1 + \frac{p}{100}$, where $p\%$ is the growth in the fourth and in the fifth year. Then

$$1.12 \cdot 1.15 \cdot 1.2 \cdot q^2 = 2.3, \quad 1.5456q^2 = 2.3, \quad q \approx 1.22$$

Production grew by about 22% in both the fourth and the fifth year.

Part II

5. a) The roots of the equation $x^2 + bx + c = 0$ are 2 greater than the roots of the equation $x^2 + cx + b = 0$. Find b and c .
 b) One root of the equation $x^2 + bx + c = 0$ is 6, and the other root is equal to the discriminant of the equation. Find b and c .
 c) The two zeros of the quadratic polynomial $x^2 + bx + c$ are x_1 and x_2 . Write a cubic polynomial whose zeros are $x_1 + x_2$, $x_1^2x_2 + x_1x_2^2$, and $x_1^3 + x_2^3$.

(16 points)

Solution: a) Let the roots of the second equation be x_1 and x_2 ; then $x_1 + x_2 = -c$ and $x_1x_2 = b$. The roots of the first equation are $x_1 + 2$ and $x_2 + 2$, so

$$(x_1 + 2) + (x_2 + 2) = -c + 4 = -b, \quad (x_1 + 2)(x_2 + 2) = b - 2c + 4 = c$$

From $b = c - 4$ and $b = 3c - 4$: $b = -4$, $c = 0$. (Indeed, $x^2 - 4x = 0$ has roots 0 and 4, and $x^2 - 4 = 0$ has roots -2 and 2.)

b) $x_1 = 6$ and $x_2 = D = b^2 - 4c$, so

$$x_1 + x_2 = 6 + b^2 - 4c = -b \quad \text{and} \quad x_1x_2 = 6(b^2 - 4c) = c$$

From $b^2 + b - 4c + 6 = 0$ and $6b^2 - 25c = 0$: $b_1 = -10$, $c_1 = 24$ (the roots are 6 and 4, and $D = 4$), and $b_2 = -15$, $c_2 = 54$ (the roots are 6 and 9, and $D = 9$).

c) Since $x_1 + x_2 = -b$ and $x_1x_2 = c$: $x_1^2x_2 + x_1x_2^2 = x_1x_2(x_1 + x_2) = -cb$, and $x_1^3 + x_2^3 = (x_1 + x_2)^3 - 3x_1x_2(x_1 + x_2) = -b^3 + 3cb$. A suitable polynomial is

$$(x + b)(x + cb)(x + b^3 - 3cb) = x^3 + b(b^2 - 2c + 1)x^2 + b^2(b^2 + b^2c - 3c^2 - 2c)x + b^3c(b^2 - 3c)$$

6. Solve the equation $(4 - \cos 8x)(2 + \cos 2x) = 15$.

(16 points)

Solution: Since the range of the cosine function is $[-1, 1]$, we have $3 \leq 4 - \cos 8x \leq 5$ and $1 \leq 2 + \cos 2x \leq 3$. So the product can be 15 only if both factors are maximal: $\cos 8x = -1$ and $\cos 2x = 1$. The first gives $x = \frac{\pi}{8} + \frac{k\pi}{4}$, and the second gives $x = n\pi$ (k, n integers). These have no common value (if $\cos 2x = 1$, then $\cos 8x = 1$), so the equation has no real solution.

7. a) In a class of 27 there are 5 more girls than boys. What is the probability that among three randomly chosen students there are two boys and one girl?
- b) We choose two students from a class of 27. Find the number of boys in the class for which the probability that the chosen students are of different sexes is as large as possible. What is this largest probability?
- c) In class 11 of 27 students, the average of a math test that went badly is exactly 2. The numbers of grades 5, 4, and 3 are equal, and we also know that every grade (from 1 to 5) occurred. What can the mode and the median of the grades be?

(16 points)

Solution: a) There are 16 girls and 11 boys. Three students can be chosen in $\binom{27}{3} = 2925$ ways, and two boys and a girl in $\binom{11}{2} \cdot 16 = 880$ ways, so the probability is $\frac{880}{2925} \approx 0.30$.

b) If there are n boys, then there are $27 - n$ girls. Two students can be chosen in $\binom{27}{2} = 351$ ways, and a boy and a girl in $n(27 - n) = 182.25 - (n - 13.5)^2$ ways. For an integer n this is largest for $n = 13$ or $n = 14$, and then the probability is $\frac{13 \cdot 14}{351} = \frac{14}{27} \approx 0.52$.

c) Let there be a grades of 1, b grades of 2, and c grades each of 3, 4, and 5. Then $a + b + 3c = 27$ and $a + 2b + 12c = 54$. Eliminating, $a = 18 - \frac{2b}{3}$ and $c = 3 - \frac{b}{9}$, so b is a multiple of 9, and $a, b, c \geq 1$: $b = 9$ or $b = 18$.

If $b = 9$: $a = 12$, $c = 2$; the mode is 1 and the median (the 14th grade) is 2.

If $b = 18$: $a = 6$, $c = 1$; the mode is 2 and the median is 2.

8. For a cubic function $f(x)$ we know that $f(1) = f(3) = 4$, the coefficient of x^3 is 1, and $\int_1^5 f(x) dx = 16$. Write the equation of the tangent line to the graph of the function at the point with x -coordinate 4.

(16 points)

Solution: Let $f(x) = x^3 + bx^2 + cx + d$. Then $f(1) = 1 + b + c + d = 4$, $f(3) = 27 + 9b + 3c + d = 4$, and

$$\int_1^5 (x^3 + bx^2 + cx + d) dx = \left[\frac{x^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx \right]_1^5 = 156 + \frac{124}{3}b + 12c + 4d = 16$$

So $b + c + d = 3$, $9b + 3c + d = -23$, $31b + 9c + 3d = -105$, whose solution is $b = -9$, $c = 23$, $d = -11$: $f(x) = x^3 - 9x^2 + 23x - 11$. Then $f'(x) = 3x^2 - 18x + 23$, $f'(4) = -1$, and $f(4) = 1$. The tangent line at $(4, 1)$ is $x + y = 5$.

9. Three vertices of the convex quadrilateral $ABCD$ are $A(2, 3)$, $B(-2, 5)$, $C(-3, -2)$, and its interior angle at the vertex D is 90° .

a) Find the area of the triangle ABC .

b) Find the coordinates of the vertex D so that the area of the quadrilateral $ABCD$ is as large as possible.

c) What is the perimeter of the quadrilateral $ABCD$, if the point D is on the x -axis?

(16 points)

Solution: a) The diagonal AC is $5\sqrt{2}$ long, and B is $3\sqrt{2}$ from the line AC (which is $y = x + 1$), so the area of ABC is $\frac{5\sqrt{2} \cdot 3\sqrt{2}}{2} = 15$.

b) The area of ABC does not change, so the area of the right triangle ACD must be as large as possible, that is, D must be as far as possible from AC . D is on the Thales circle of AC (center $\left(-\frac{1}{2}, \frac{1}{2}\right)$), on the side of AC opposite

B , so it is the intersection of this circle with the perpendicular bisector $x + y = 0$ of AC : $D(2, -2)$. (Then the area of $ABCD$ is $15 + 12.5 = 27.5$.)

c) Let $D(d, 0)$. Then $AD^2 = (d - 2)^2 + 9$ and $CD^2 = (d + 3)^2 + 4$, and since ACD is a right triangle, $(d - 2)^2 + 9 + (d + 3)^2 + 4 = 50$, that is, $d^2 + d - 12 = 0$. Of its roots only $d = 3$ gives a convex quadrilateral ($d = -4$ is on the same side of AC as B). The perimeter of $ABCD$ with $D(3, 0)$ is

$$AB + BC + CD + DA = 2\sqrt{5} + 5\sqrt{2} + 2\sqrt{10} + \sqrt{10} \approx 21.03$$