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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. Write the equation of the circle that passes through the point $(5, 5)$ and touches the circle $(x + 2)^2 + (y - 4)^2 = 10$ at the point $(1, 3)$.

(11 points)

Solution: The center of the given circle is $(-2, 4)$. The center of the circle we are looking for is on the line through $(1, 3)$ and $(-2, 4)$, which is $x + 3y = 10$. It is also on the perpendicular bisector of the chord joining $(1, 3)$ and $(5, 5)$, which is $2x + y = 10$. The two lines meet at $(4, 2)$, the center. The radius is the distance of $(1, 3)$ and $(4, 2)$, $\sqrt{10}$, so the circle is

$$(x - 4)^2 + (y - 2)^2 = 10$$

2. Candies of the brand “Sajó Söder” are packed in fifties. Quality control found that a package contains exactly 50 candies only with probability 0.9.

- What is the probability that, buying six packages of candies, every package contains 50 candies?
- What is the probability that among the six packages there are at least two that do not contain 50 candies?

(12 points)

Solution: We assume that the packages are chosen at random, independently of each other.

- $0.9^6 \approx 0.531$.
- The complement is that at most one package is bad, so the probability is $1 - (0.9^6 + 6 \cdot 0.9^5 \cdot 0.1) \approx 0.114$.

3. Find the set $A \cap B$, if $A = \{x \in \mathbb{R} \mid 3(\cot x - \tan x) = 2\sqrt{3}\}$ and $B = \{x \in \mathbb{R} \mid |x| \leq 2\}$.

(14 points)

Solution: The equation $3(\cot x - \tan x) = 2\sqrt{3}$ (with $x \neq \frac{k\pi}{2}$) can be written as $3\tan^2 x + 2\sqrt{3}\tan x - 3 = 0$, so

$$\tan x = \frac{\sqrt{3}}{3} \text{ or } \tan x = -\sqrt{3}:$$

$$x = \frac{\pi}{6} + k\pi \quad \text{or} \quad x = -\frac{\pi}{3} + n\pi, \quad k, n \text{ integers}$$

Of these, $-2 \leq x \leq 2$ holds only for $x = \frac{\pi}{6}$ and $x = -\frac{\pi}{3}$, so $A \cap B = \left\{-\frac{\pi}{3}, \frac{\pi}{6}\right\}$.

4. About three different right circular cones we know that both the radii of their bases and their heights are three consecutive terms of an arithmetic sequence, with the same difference. Show that the volumes of the cones cannot be three consecutive terms of an arithmetic sequence.

(14 points)

Solution: Let the radii be $r - d, r, r + d$ and the heights $m - d, m, m + d$, where $d \neq 0$ since the cones are different.

The volumes are $V_1 = \frac{(r-d)^2 \pi (m-d)}{3}$, $V_2 = \frac{r^2 \pi m}{3}$, $V_3 = \frac{(r+d)^2 \pi (m+d)}{3}$. They would be consecutive terms of an arithmetic sequence only if $V_3 - V_2 = V_2 - V_1$. But

$$V_3 - V_2 = \frac{\pi}{3} (d^3 + r^2 d + 2rd^2 + 2rmd + md^2)$$

$$V_2 - V_1 = \frac{\pi}{3} (d^3 + r^2 d - 2rd^2 + 2rmd - md^2)$$

and their difference is $\frac{\pi}{3} (4rd^2 + 2md^2) > 0$, since $d \neq 0$ and $r, m > 0$. So the volumes cannot form an arithmetic sequence.

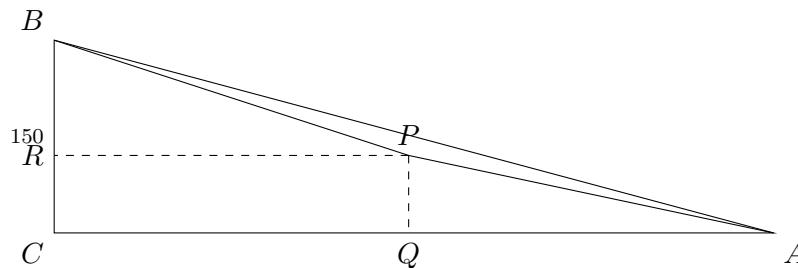
Part II

5. The shorter leg of a right triangle is 150 units, and one of its acute angles is 15° . We join an interior point P of the triangle with the two endpoints of the hypotenuse. These segments and the two legs are the sides of a concave quadrilateral whose acute angles are 12° and 72° .

- Which endpoint of the hypotenuse is P closer to?
- How far is P from the longer leg?
- What is the area of this concave quadrilateral?

(16 points)

Solution: Let the right angle be at C , $BC = 150$, $\angle CAB = 15^\circ$, $\angle ABC = 75^\circ$; the acute angles of the quadrilateral $ACBP$ are $\angle CAP = 12^\circ$ and $\angle PBC = 72^\circ$. Let Q and R be the projections of P onto CA and CB .



- $\angle PAB = 15^\circ - 12^\circ = 3^\circ$ and $\angle ABP = 75^\circ - 72^\circ = 3^\circ$ (so $\angle APB = 174^\circ$). The triangle ABP has two equal angles, so $AP = BP$: P is equally far from the two endpoints of the hypotenuse.

- In the triangle ABC , $AB = \frac{150}{\sin 15^\circ}$. By the Law of Sines in the triangle APB , $\frac{AP}{AB} = \frac{\sin 3^\circ}{\sin 174^\circ}$, so

$$AP = \frac{150}{\sin 15^\circ} \cdot \frac{\sin 3^\circ}{\sin 174^\circ} \approx 290.2$$

In the right triangle APQ , $PQ = AP \sin 12^\circ \approx 60.3$: this is the distance of P from the longer leg CA .

- $AC = \frac{150}{\tan 15^\circ} \approx 559.8$. The area of the quadrilateral is the area of ABC minus the area of APB :

$$t = \frac{150 \cdot 559.8}{2} - \frac{290.2^2 \cdot \sin 174^\circ}{2} \approx 41\,985 - 4401 \approx 37\,585 \text{ square units}$$

6. Consider the parabolas with equation $y = (p - 1)x^2 + 2px + 4$, where p is any real number different from 1.
- Is there a value of the parameter p for which the parabola has no common point with the x -axis?
 - Find the points that lie on every member of the family of parabolas above.
 - Find the value of the parameter p for which the sum of the squares of the roots of the equation $(p - 1)x^2 + 2px + 4 = 0$ is 5.

(16 points)

Solution: a) The discriminant of $(p - 1)x^2 + 2px + 4 = 0$ is $4p^2 - 16(p - 1) = 4(p - 2)^2 \geq 0$, so every parabola has a common point with the x -axis: there is no such p .

b) Arranging by powers of p : $y = (x^2 + 2x)p - (x^2 - 4)$. y does not depend on p exactly when $x^2 + 2x = 0$, that is, $x = 0$ or $x = -2$. So every parabola passes through $(0, 4)$ and $(-2, 0)$, and there are no other such points.

c) By Viète's formulas

$$x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1x_2 = \left(\frac{-2p}{p-1}\right)^2 - \frac{8}{p-1} = \frac{4p^2 - 8p + 8}{(p-1)^2} = 5$$

so $p^2 - 2p - 3 = 0$, and $p = -1$ or $p = 3$ (the roots are real for every p , by part a).

7. Find the integers a and b for which $\lg(3a + 2b) = \lg a + \lg b$. ($\lg$ is the base 10 logarithm.)

(16 points)

Solution: Because of the logarithms, $a > 0$ and $b > 0$. The equation is $\lg(3a + 2b) = \lg ab$, that is, $3a + 2b = ab$, or $(a - 2)(b - 3) = 6$. The possibilities with $a, b > 0$ are $(a - 2, b - 3) = (1, 6), (2, 3), (3, 2), (6, 1)$ (both factors cannot be negative: then $a < 2$, so $a = 1$, $a - 2 = -1$, and $b = -3 < 0$). The solutions are $(a, b) = (3, 9), (4, 6), (5, 5), (8, 4)$.

8. We rotate the graph of the function $f(x) = \sqrt{x}$, defined on the interval $[0, a]$, around the x -axis.
- Find the largest integer a for which the volume of the resulting solid of revolution does not exceed 1000 units.
 - Write the equation of the tangent line to the graph of f at the point with x -coordinate 4.

(16 points)

Solution: a) For $a > 0$:

$$V = \pi \int_0^a (\sqrt{x})^2 dx = \pi \left[\frac{x^2}{2} \right]_0^a = \frac{a^2 \pi}{2} \leq 1000$$

gives $a \leq \sqrt{\frac{2000}{\pi}} \approx 25.2$, so the largest such integer is 25.

b) $f'(x) = \frac{1}{2\sqrt{x}}$, so the slope of the tangent at $x = 4$ is $\frac{1}{4}$, and the point of tangency is $(4, 2)$. The tangent line is $y - 2 = \frac{1}{4}(x - 4)$, that is, $y = \frac{x}{4} + 1$.

9. Find the acute angle for which the sum $4 \cos^2 x + \frac{1}{\cos^2 x}$ is minimal. What is this smallest value?

(16 points)

Solution: For an acute angle x both $4 \cos^2 x$ and $\frac{1}{\cos^2 x}$ are defined and positive. By the inequality between the arithmetic and geometric means,

$$4 \cos^2 x + \frac{1}{\cos^2 x} \geq 2 \sqrt{4 \cos^2 x \cdot \frac{1}{\cos^2 x}} = 4$$

with equality exactly when $4 \cos^2 x = \frac{1}{\cos^2 x}$, that is, $\cos x = \frac{\sqrt{2}}{2}$, so $x = \frac{\pi}{4}$. The smallest value is 4, at $x = 45^\circ$.