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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. The vertices of the triangle ABC are $A(-2, -3)$, $B(8, 1)$, and $C(2, 5)$.

- Write the equations of the lines of the sides of the triangle.
- Compute the largest interior angle of the triangle.

(11 points)

Solution: a) A direction vector of the line a of the side BC is $(3, -2)$, and its equation is $2x + 3y = 19$. A direction vector of the line b of AC is $(1, 2)$, and its equation is $2x - y = -1$. A direction vector of the line c of AB is $(5, 2)$, and its equation is $2x - 5y = 11$.

b) Since $AB = \sqrt{116}$, $BC = \sqrt{52}$, and $AC = \sqrt{80}$, the largest angle is γ , opposite AB . By the Law of Cosines (or using $\vec{CA} \cdot \vec{CB} = 8$),

$$\cos \gamma = \frac{8}{\sqrt{52} \cdot \sqrt{80}} \approx 0.1240, \quad \text{so} \quad \gamma \approx 82.9^\circ$$

2. a) Decide which of the following statements are true and which are false.

- A quadrilateral can have two parallel side lines.
- A quadrilateral can have two opposite side lines that are parallel.
- A quadrilateral can have two neighboring side lines that are parallel.
- A quadrilateral can have two perpendicular side lines.
- A quadrilateral can have two opposite side lines that are perpendicular.
- A quadrilateral can have two neighboring side lines that are perpendicular.

b) If someone writes the true or false answers in the previous problem at random, what is the probability that all of the answers are correct?

c) In problem a) it is also possible that someone cannot decide, and writes nothing for some statements. How many different ways of answering are possible in all?

(12 points)

Solution: a) 1) true, 2) true, 3) false (two neighboring sides have a common vertex, so their lines cannot be parallel), 4) true, 5) true, 6) true.

b) There are $2^6 = 64$ equally likely ways of answering, and one of them is correct: the probability is $\frac{1}{64} \approx 0.016$.

c) Each answer can be true, false, or empty, so there are $3^6 = 729$ ways.

3. For which x are the following expressions defined?

a) $\lg \sqrt{x^2 + 4x - 12}$ b) $\sqrt{\lg(x^2 + 4x - 12)}$

(lg is the base 10 logarithm.)

(14 points)

Solution: a) Because of the square root, $x^2 + 4x - 12 = (x + 6)(x - 2) \geq 0$, and because of the logarithm, $\sqrt{x^2 + 4x - 12} > 0$. So $(x + 6)(x - 2) > 0$, that is,

$$x \in (-\infty, -6) \cup (2, \infty)$$

b) Because of the logarithm, $x^2 + 4x - 12 > 0$, and because of the square root, $\lg(x^2 + 4x - 12) \geq 0 = \lg 1$. The base 10 logarithm is strictly increasing, so $x^2 + 4x - 12 \geq 1$, that is, $x^2 + 4x - 13 \geq 0$ (this condition is the stronger one):

$$x \in \left(-\infty, -2 - \sqrt{17}\right] \cup \left[-2 + \sqrt{17}, \infty\right)$$

4. Let $f(x) = x^6 - 3x^4 + 3x^2$. Compute the value $f\left(\sqrt{\sqrt[3]{3} + 1}\right)$.

(14 points)

Solution: With $y = x^2$ we need $g(y) = y^3 - 3y^2 + 3y$, where $y = \sqrt[3]{3} + 1$. Since $g(y) = (y - 1)^3 + 1$, the value is $(\sqrt[3]{3})^3 + 1 = 4$.

Part II

5. In the coordinate system we join the origin with the point (a, a^2) (a a positive real number) in two ways: first with the line through the two points, then with the graph of the function $f(x) = x^2$ through the two points. In both cases we rotate the connecting curve around the x -axis.

a) Compute the volumes of the two solids of revolution obtained this way, if $a = 2$.

b) Find, as a function of a , what percent of the volume of the larger solid is the volume of the smaller solid.

(16 points)

Solution: a) In the first case we get a right circular cone with base radius 4 and height 2; its volume is $V_1 = \frac{16\pi \cdot 2}{3} = \frac{32\pi}{3} \approx 33.51$. The volume of the second solid is

$$V_2 = \pi \int_0^2 x^4 dx = \pi \left[\frac{x^5}{5} \right]_0^2 = \frac{32\pi}{5} \approx 20.11$$

b) In general the cone has base radius a^2 and height a , so $V_1 = \frac{a^4\pi \cdot a}{3} = \frac{a^5\pi}{3}$, and $V_2 = \pi \int_0^a x^4 dx = \frac{a^5\pi}{5}$. So $\frac{V_2}{V_1} = \frac{3}{5} = 0.6$: the smaller solid is 60% of the larger one, independently of a .

6. Find the real numbers x for which

$$\frac{\sqrt{x+5}-2}{\sqrt{x+7}+3} : \frac{\sqrt{x+6}-1}{\sqrt{x+8}+4} \leq 0$$

(16 points)

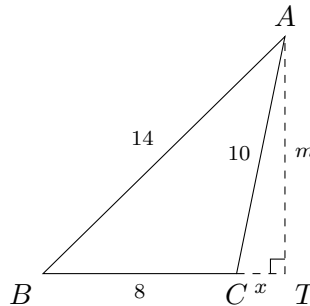
Solution: Because of the square roots, $x \geq -5$. Because of the division, the numerator of the second fraction cannot be 0, so $x \neq -5$: the domain is $(-5, \infty)$. Both denominators are positive, so the inequality holds exactly when $(\sqrt{x+5}-2)(\sqrt{x+6}-1) \leq 0$. Multiplying by the positive number $(\sqrt{x+5}+2)(\sqrt{x+6}+1)$: $(x+1)(x+5) \leq 0$, that is, $-5 \leq x \leq -1$. Taking the domain into account, the solution is $-5 < x \leq -1$.

(Shorter: for $x > -5$ the factor $\sqrt{x+6}-1$ is positive, so the condition is $\sqrt{x+5} \leq 2$.)

7. a) Compute the volume of the right prism with a triangular base whose base edges are $a = 8$ cm, $b = 10$ cm, $c = 14$ cm, if we know that the height of the prism is $m = m_a$, where m_a is the altitude of the base triangle to the side a .
b) What is the volume of the largest cylinder that fits in the prism above and whose axis is parallel to the height of the prism?

(16 points)

Solution: a) Let $BC = 8$, $CA = 10$, $AB = 14$. Since $AB^2 > BC^2 + AC^2$, the angle ACB is obtuse, so the altitude from A falls outside the triangle; let its foot be T , and $CT = x$:



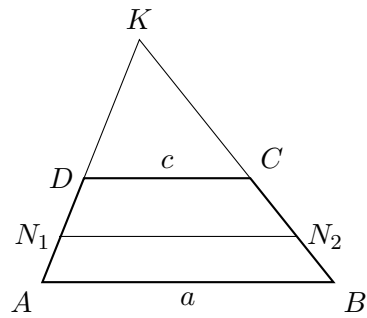
By the Pythagorean Theorem in the triangles ABT and ACT : $(8+x)^2 + m^2 = 196$ and $x^2 + m^2 = 100$. So $64 + 16x + 100 = 196$, $x = 2$ cm, and $m = \sqrt{96} = 4\sqrt{6}$ cm. The area of the base is $t = \frac{8 \cdot 4\sqrt{6}}{2} = 16\sqrt{6}$ cm², and the volume of the prism is $V_1 = tm = 16\sqrt{6} \cdot 4\sqrt{6} = 384$ cm³.

b) The radius of the base of the cylinder is the radius ϱ of the circle inscribed in the triangle ABC . The semiperimeter is $s = 16$ cm, and from $t = \varrho s$, $\varrho = \frac{16\sqrt{6}}{16} = \sqrt{6}$ cm. The volume of the cylinder is $V_2 = \varrho^2 \pi m = 6\pi \cdot 4\sqrt{6} = 24\sqrt{6}\pi \approx 184.7$ cm³.

8. We want to divide a trapezoid-shaped plot into two parts of equal area with a fence that is parallel to the parallel sides and joins a point of each leg. Give the length of the fence as a function of the lengths of the two parallel sides of the trapezoid.

(16 points)

Solution: Let the parallel sides be $AB = a$ and $DC = c$ ($a > c$), let the lines of the legs meet at K , and let the fence be N_1N_2 . The triangles KDC , KN_1N_2 , KAB are similar, so their areas are proportional to the squares of their sides opposite K : there is a positive λ such that the three areas are λc^2 , $\lambda \cdot N_1N_2^2$, λa^2 .



Since the fence bisects the area of the trapezoid, the area of the middle triangle is the arithmetic mean of the other two:

$$\lambda \cdot N_1 N_2^2 = \frac{\lambda a^2 + \lambda c^2}{2}. \text{ So}$$

$$N_1 N_2 = \sqrt{\frac{a^2 + c^2}{2}}$$

(the root mean square of the parallel sides).

9. Find the values of the parameter k for which both of the equations

$$\sin^4 x + \cos^4 x = k - \frac{1}{4} \quad \text{and} \quad \sin^4 x + \cos^4 x = k + \frac{1}{4}$$

have a solution.

(16 points)

Solution: $\sin^4 x + \cos^4 x = (\sin^2 x + \cos^2 x)^2 - 2 \sin^2 x \cos^2 x = 1 - \frac{1}{2} \sin^2 2x$, and since $0 \leq \sin^2 2x \leq 1$, this expression takes exactly the values in $\left[\frac{1}{2}, 1\right]$. So the first equation has a solution if $\frac{1}{2} \leq k - \frac{1}{4} \leq 1$, that is, $\frac{3}{4} \leq k \leq \frac{5}{4}$, and the second if $\frac{1}{2} \leq k + \frac{1}{4} \leq 1$, that is, $\frac{1}{4} \leq k \leq \frac{3}{4}$. Both have a solution only for $k = \frac{3}{4}$.