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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

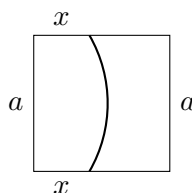
1. From a container holding 50 liters of ethyl alcohol of 80% concentration we took out some, and replaced the missing amount with water. Then we took out 3 liters less than before, and replaced this amount with water too. Now the container holds equal amounts of alcohol and water. How much did we take out the first time?

Solution: The container originally held $50 \cdot 0.8 = 40$ liters of alcohol. After taking out x liters, $40 - 0.8x$ liters of alcohol remain, and after filling up with water, the concentration is $\frac{40 - 0.8x}{50}$. The second time we take out $(x - 3) \cdot \frac{40 - 0.8x}{50}$ liters of alcohol, so

$$40 - 0.8x - (x - 3) \cdot \frac{40 - 0.8x}{50} = 25$$

Rearranging, we get a quadratic equation, and only one of its roots makes sense: $x \approx 11.94$ liters.

2. What is x , if the arc of radius a bisects the area of the square with side a ? (The arc joins the endpoints of the two segments of length x on the top and bottom sides; it bulges to the right, and its center is outside the square, on the left.)



Solution: The arc joins the points at distance x from the left side on the top and the bottom side, so its chord is a vertical segment of length a ; since the radius is also a , the central angle of the arc is 60° . The part of the square to the left of the arc consists of an $a \times x$ rectangle and a circular segment of area

$$T = \frac{a^2\pi}{6} - \frac{a^2\sqrt{3}}{4}$$

(a 60° sector minus an equilateral triangle). So $ax + \frac{a^2\pi}{6} - \frac{a^2\sqrt{3}}{4} = \frac{a^2}{2}$, that is,

$$x = a \left(\frac{1}{2} + \frac{\sqrt{3}}{4} - \frac{\pi}{6} \right) \approx 0.41a$$

3. We released 379 bees in a room in the shape of a rectangular box with edges 9 m, 7 m, 6 m. Prove that at any moment there are 2 bees whose distance is less than 1.8 m.

Solution: The volume of the room is 378 m^3 . Divide it into 378 unit cubes of volume 1 m^3 , and consider the bees as points. Assign each bee to a cube that contains it (a bee on a boundary is assigned to one of the cubes whose boundary it is on). By the pigeonhole principle there is a cube with at least two bees. Their distance is at most the space diagonal of the unit cube, $\sqrt{3} \approx 1.732 \text{ m}$, so less than 1.8 m.

(The KoMaL solution is written for a variant with 41 bees in a $5 \times 4 \times 2$ room; the argument is the same.)

4. a) Winnie the Pooh took out a loan of 10 000 coins from the Ant Savings Cooperative for 10 years, at 8% yearly interest, with yearly repayment, to buy the neighboring raspberry patch. How much does Pooh have to pay back each year? (The repayment installment is the same every year.)

b) For how many years should Pooh have taken out the loan, if he wanted to pay back 2200 coins a year?

Solution: a) Paying back the installment x at the end of each year, after 10 years the debt is

$$10\,000 \cdot 1.08^{10} - x \cdot 1.08^9 - x \cdot 1.08^8 - \dots - x \cdot 1.08 - x = 10\,000 \cdot 1.08^{10} - x \cdot \frac{1.08^{10} - 1}{1.08 - 1} = 0$$

so $x \approx 1490.3$ coins.

b) Similarly, $10\,000 \cdot 1.08^n - 2200 \cdot \frac{1.08^n - 1}{1.08 - 1} = 0$, whose solution is $n \approx 5.87$. So the loan should have been taken out for at least 6 years (with a smaller last installment).

(The KoMaL solution is written for a variant of the story with a car loan of 10 000 euros; the numbers are the same.)

Part II

5. a) Find the real solutions of the equation $\sin 2x + 1 = 3(\sin x + \cos x) - 2$.

b) For which natural numbers n is the expression $3 \cdot 5^n + n^2 + 4n + 4$ divisible by 100?

Solution: a) Since $\sin 2x + 1 = 2 \sin x \cos x + \sin^2 x + \cos^2 x = (\sin x + \cos x)^2$, the equation is

$$(\sin x + \cos x)^2 - 3(\sin x + \cos x) + 2 = 0$$

so $\sin x + \cos x = 2$ (impossible, since $\sin x + \cos x \leq \sqrt{2}$) or $\sin x + \cos x = 1$, which gives $x = 2k\pi$ or $x = \frac{\pi}{2} + 2k\pi$ (k an integer).

b) For $n = 0$ and $n = 1$ the expression is not divisible by 100. For $n \geq 2$, 5^n ends in 25, so $3 \cdot 5^n$ ends in 75, and the expression is divisible by 100 exactly when $n^2 + 4n + 4 = (n + 2)^2$ ends in 25. The square of a natural number ends in 25 exactly when the number ends in 5. So $n + 2$ ends in 5: $n = 10k + 3$, where k is a nonnegative integer.

6. The following functions are given:

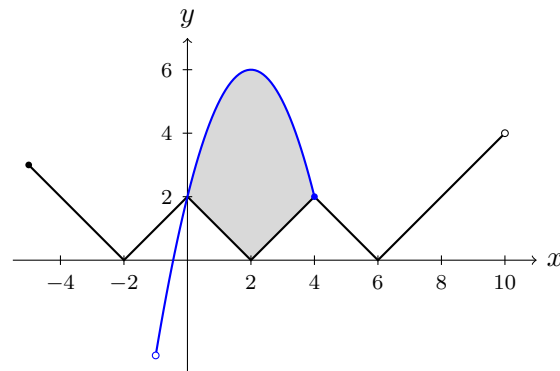
$$f : x \in [-5, 10], \quad x \mapsto ||x - 2| - 2| - 2|, \quad g : x \in (-1, 4], \quad x \mapsto -x^2 + 4x + 2$$

a) Graph the functions f and g in the same coordinate system.

b) For which x is $f(x) > g(x)$?

c) Find the area of the region bounded by the two graphs.

Solution: a) The graph of f is a zigzag line through $(-5, 3)$, $(-2, 0)$, $(0, 2)$, $(2, 0)$, $(4, 2)$, $(6, 0)$, up to (but not including) $(10, 4)$; the graph of g is a part of a parabola with vertex $(2, 6)$:



b) On the common domain $(-1, 4]$: $f(x) > g(x)$ for $-1 < x < 0$.

c) The two graphs meet at $(0, 2)$ and $(4, 2)$, and the region between them (shaded) is cut into two parts by the line $y = 2$: above it, the region under the parabola; below it, the triangle with vertices $(0, 2)$, $(2, 0)$, $(4, 2)$. So

$$T = \frac{4 \cdot 2}{2} + \int_0^4 (-x^2 + 4x + 2 - 2) dx = 4 + \left[-\frac{x^3}{3} + 2x^2 \right]_0^4 = 4 + \frac{32}{3} = \frac{44}{3}$$

7. Student cartographers recorded, in a 3-dimensional coordinate system, the data of a square terrain of size 1000 m by 1000 m seen from above. The origin is the southwest corner of the area, whose height above sea level is 320 m. The coordinates of the terrain points are in meters. The first coordinate (x) is the extent in the east direction, the second (y) in the north direction. The vertical extent is given by the third coordinate (z) of the triples (z can also be negative).

a) At what angle of elevation is the peak B $(221, 52, 340)$ seen from the peak A $(31, 40, 110)$?

b) What is the equation, in the x, y coordinate system, of the perpendicular projection of the line joining the peaks A and B onto the plane determined by the x and y axes?

c) What is the height above sea level of the rock whose peak C , together with the points

$$D(400, 400, 30), \quad E(500, 500, 30), \quad F(500, 400, 30)$$

determines a pyramid of volume $100\,000 \text{ m}^3$?

Solution: a) The segment AB is the hypotenuse of a right triangle whose vertical leg is $340 - 110 = 230 \text{ m}$ long, and whose other leg is the distance of the projections of A and B onto the xy -plane: $\sqrt{(221 - 31)^2 + (52 - 40)^2} \approx 190.379 \text{ m}$. So $\tan \alpha = \frac{230}{190.379}$, and $\alpha \approx 50.38^\circ$.

b) The line through $A'(31, 40)$ and $B'(221, 52)$ has direction vector $(190, 12)$ and normal vector $(-12, 190)$, so its equation is $-12x + 190y = 7228$.

c) D, E, F have the same z -coordinate, so they determine a horizontal triangle: an isosceles right triangle with legs 100 m and area 5000 m^2 . From $100\,000 = \frac{5000 \cdot m}{3}$, the height of the pyramid is $m = 60 \text{ m}$. So the height of C above sea level is $320 + 30 + 60 = 410 \text{ m}$ or $320 + 30 - 60 = 290 \text{ m}$.

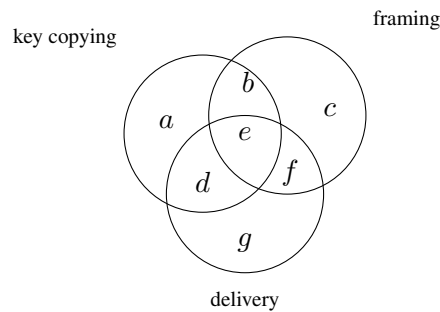
8. In a large city there are 21 service centers. Of these, thirteen do key copying, eleven do picture framing, and eight deliver the finished work to the customer's home. Every service center offers at least one of these three services. There are four service centers that do no picture framing and no home delivery. Three service centers do not deliver, but offer the other two services. There is only one service center that delivers to the home but does not offer the other two services.

a) Show that there are no two service centers that offer all three services.

b) How many service centers do only picture framing (of the three services)?

c) We hear that next week one of the service centers will add a new one of these three services to its profile. What is the probability that after this there will be two service centers offering all three services?

Solution: Let $a, b, \dots, g$ be the number of service centers in the regions of the diagram:



a) From the text, $a = 4$, $b = 3$, $g = 1$, and

$$a + b + c + d + e + f + g = 21, \quad a + b + d + e = 13, \quad b + c + e + f = 11, \quad d + e + f + g = 8$$

Solving the system, $e = 1$ (and $c = 6$, $d = 5$, $f = 1$): only one service center offers all three services.

b) $c = 6$.

c) The 20 service centers that do not offer all three services can extend their profile; these are the elementary events. The favorable ones are the centers that offer exactly two services: $b + d + f = 3 + 5 + 1 = 9$. The probability is $\frac{9}{20}$. (In the KoMaL solution the third service is called shoe repair instead of home delivery; the numbers are the same.)

9. The following table contains some data of a group of 12 students of a sports school:

Number in the register	Height (cm)	Shoe size	Long jump best in 2004 (cm)	Long jump best in 2005 (cm)
1.	169	40	401	429
2.	180	42	432	425
3.	159	40	370	405
4.	191	45	520	531
5.	181	43	462	461
6.	176	44	448	482
7.	174	43	430	422
8.	170	42	419	426
9.	173	44	359	380
10.	188	44	602	581
11.	201	47	—	473
12.	169	40	388	388

- a) Compute by what percent the students' personal bests in long jump changed from 2004 to 2005. List the students' numbers in the register in increasing order of this change.
- b) Show the frequency and the relative frequency of the shoe sizes in suitable diagrams.
- c) In how many ways can three students be chosen so that among them there is one shorter than the average and one taller than the average?

Solution: a) The changes (for student 11 the 2004 data is missing, so we cannot compute it):

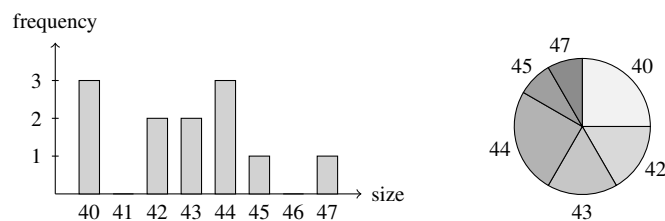
student	1	2	3	4	5	6	7	8	9	10	12
change	6.98%	-1.62%	9.46%	2.12%	-0.22%	7.59%	-1.86%	1.67%	5.85%	-3.49%	0.00%

In increasing order of the change: 10, 7, 2, 5, 12, 8, 4, 9, 1, 6, 3.

- b) The frequencies and relative frequencies of the shoe sizes:

size	40	41	42	43	44	45	46	47
frequency	3	0	2	2	3	1	0	1
relative frequency	25%	0%	16.67%	16.67%	25%	8.33%	0%	8.33%

For example, a bar chart of the frequencies and a pie chart of the relative frequencies:



- c) The average height is $\frac{2131}{12} \approx 177.58$ cm; 7 students are shorter and 5 are taller. Three students can be chosen in $\binom{12}{3} = 220$ ways; of these, $\binom{7}{3} = 35$ consist of shorter students only, and $\binom{5}{3} = 10$ of taller students only. So in $220 - 35 - 10 = 175$ cases there is both a shorter and a taller student among the three.