

author: Zoltán Csík

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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. The difference of the sums of the interior angles of two convex polygons is equal to the sum of the interior angles of a heptagon, and one of them has twice as many vertices as the other has sides. How many diagonals do they have together?

Solution: Let the polygons have n and m vertices. The sums of their interior angles are $(n - 2) \cdot 180^\circ$ and $(m - 2) \cdot 180^\circ$, and their difference is $(n - m) \cdot 180^\circ = (7 - 2) \cdot 180^\circ$, so $n - m = 5$. A polygon has as many vertices as sides, so $n = 2m$. So $n = 10$ and $m = 5$, and the number of diagonals is

$$\frac{10 \cdot 7}{2} + \frac{5 \cdot 2}{2} = 35 + 5 = 40$$

2. Solve the following system of equations over the pairs of positive integers:

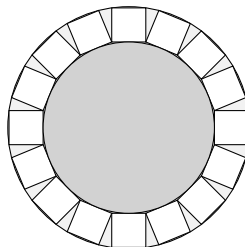
$$x^{y^2-15y+56} = 1, \quad y - x = 5$$

Solution: For a positive integer x , the first equation holds if the exponent $y^2 - 15y + 56 = (y - 7)(y - 8)$ is 0, that is, $y = 7$ or $y = 8$; or if $x = 1$. Using the second equation, the solutions are $(2, 7)$, $(3, 8)$, and $(1, 6)$.

3. The roots of the equation $k^2(2 + 4x) - x(x + 1 + k) + 3(x - 2)(2k + 1) - (1 - x)(k + 3) = 0$ are $x_1 = 1$ and $x_2 = 2$. Find the value of k .

Solution: Substituting $x_1 = 1$, we must get 0: $6k^2 - (2 + k) - 3(2k + 1) = 0$, that is, $6k^2 - 7k - 5 = 0$, so $k = \frac{5}{3}$ or $k = -\frac{1}{2}$. Similarly, substituting $x_2 = 2$: $10k^2 - 2(3 + k) + (k + 3) = 0$, that is, $10k^2 - k - 3 = 0$, so $k = \frac{3}{5}$ or $k = -\frac{1}{2}$. Both conditions must hold, so $k = -\frac{1}{2}$.

4. In a tea house, tea is served so that 16 sugar cubes can be placed around the cup as shown in the figure (the corners of neighboring sugar cubes touch, each touches the cup with one face, and two of its vertices are on the rim of the saucer). How many cm are the edge of the sugar cube and the radius of the cup, if the diameter of the saucer is 10 cm?



Solution: Each sugar cube belongs to a central angle of $\frac{2\pi}{16}$. Let x be the edge of the cube, r the radius of the cup, and $R = 5$ cm the radius of the saucer. The inner face of the cube touches the cup, and its two inner corners are on the boundaries of the central angle, so $r = \frac{x}{2} \cot \frac{\pi}{16}$. The outer corners are on the rim of the saucer, so

$$R^2 = (x + r)^2 + \left(\frac{x}{2}\right)^2$$

With $\cot \frac{\pi}{16} \approx 5.027$ we get $25 = x^2 \left((1 + 2.514)^2 + 0.25 \right)$, so $x \approx 1.41$ cm, and $r \approx 3.54$ cm. The edge of the sugar cube is about 1.4 cm, and the cup has radius about 3.5 cm (diameter about 7 cm).

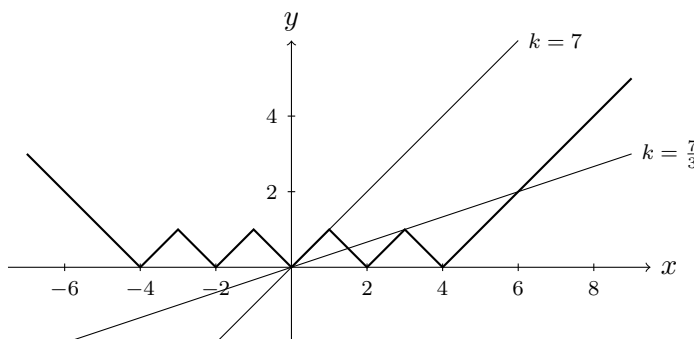
Part II

5. How many solutions does the equation

$$|||x| - 1| - 1| - 1| - 1| = \frac{kx}{7}$$

have, depending on the positive integer k ?

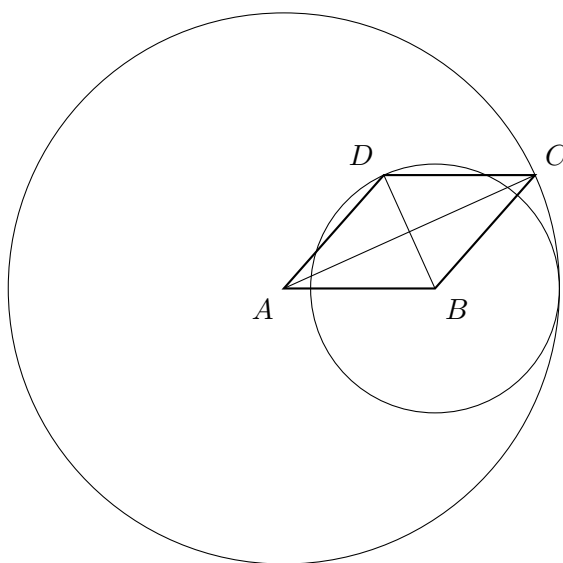
Solution: The graph of $f(x) = |||x| - 1| - 1| - 1| - 1|$ is a zigzag line through $(0, 0)$, $(\pm 1, 1)$, $(\pm 2, 0)$, $(\pm 3, 1)$, $(\pm 4, 0)$, and for $|x| \geq 4$ it is $|x| - 4$. We intersect it with the line $y = \frac{k}{7}x$ through the origin:



The boundary cases are the slope 1 (the line $y = x$ contains the part of the graph for $x \geq 4$) and the slope $\frac{1}{3}$ (the line through $(3, 1)$). Counting the intersections: if $k > 7$, there is one solution ($x = 0$); if $k = 7$, infinitely many; if $k \in \{3, 4, 5, 6\}$, three; and if $k \in \{1, 2\}$, five. (We checked these numbers by computer.)

6. $ABCD$ is a rhombus with side 10 units. The circle centered at A and passing through the vertex C is touched from inside by the circle centered at B and passing through the vertex D . Find the area of the rhombus.

Solution: Let $R = AC$ be the radius of the larger circle (the longer diagonal of the rhombus) and $r = BD$ the radius of the smaller circle (the shorter diagonal).

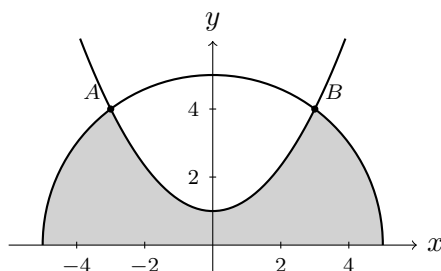


The diagonals of a rhombus bisect each other at right angles, so by the Pythagorean Theorem $\frac{R^2}{4} + \frac{r^2}{4} = 100$. The circles touch internally, so $R - r = AB = 10$. From these, $2r^2 + 20r - 300 = 0$, whose positive root is $r = 5\sqrt{7} - 5$, and then $R = 5\sqrt{7} + 5$. The area of the rhombus is

$$t = \frac{Rr}{2} = \frac{(5\sqrt{7} + 5)(5\sqrt{7} - 5)}{2} = 75$$

7. The circle $x^2 + y^2 = 25$, the parabola $y = \frac{1}{3}x^2 + 1$, and the x -axis determine a plane region, which we rotate around the x -axis. What is the volume of the resulting solid of revolution?

Solution: The parabola and the circle meet at $A(-3, 4)$ and $B(3, 4)$. The region (shaded) lies between the x -axis and the parabola for $|x| \leq 3$, and between the x -axis and the circle for $3 \leq |x| \leq 5$:



$$\begin{aligned} V &= 2\pi \left[\int_0^3 \left(\frac{1}{3}x^2 + 1 \right)^2 dx + \int_3^5 (25 - x^2) dx \right] = 2\pi \left[\frac{x^5}{45} + \frac{2x^3}{9} + x \right]_0^3 + 2\pi \left[25x - \frac{x^3}{3} \right]_3^5 \\ &= \frac{144}{5}\pi + \frac{104}{3}\pi = \frac{952}{15}\pi \approx 199.4 \end{aligned}$$

8. A company produced solid spherical chewing gum and sold it for 10 forints apiece. Because the raw materials became more expensive, the production cost increased by 300%. The manufacturer would like the ratio of profit to production cost after the sale of the gum not to change, so it raised the price of the gum to 20 forints apiece. The product was also redesigned: the spherical chewing gum of diameter 2 cm now has a concentric spherical cavity inside. Compute the thickness of the wall of the new chewing gum.

Solution: Let the production cost of the old gum be x Ft, and let y be the ratio of the amount (mass) of material in the new and the old gum. The material is now 4 times as expensive, so the new gum costs $4xy$ Ft to produce. The profit per piece is $10 - x$ for the old gum and $20 - 4xy$ for the new one. The ratio of profit to cost did not change:

$$\frac{10 - x}{x} = \frac{20 - 4xy}{4xy}, \quad y = \frac{1}{2}$$

So the new gum uses half as much material as the old one: the cavity has half the volume of the ball of radius 1 cm, so its radius is $\sqrt[3]{\frac{1}{2}}$ cm. The thickness of the wall is $1 - \sqrt[3]{\frac{1}{2}} \approx 0.2$ cm.

9. Solve the following systems of equations:

a) $\sqrt{x-4} + \sqrt{x-5} - 1 = 7x - x^2 - 2xy, \quad \log_5 64^x \cdot \log_4 125^y = 45;$

b) $xy + y = 2, \quad (1-y) \cdot 2 \cos \frac{x\pi}{3} = \frac{x-1}{x+1}.$

Solution: a) We need $x \geq 5$. The left-hand side of the second equation is

$$\log_5 4^{3x} \cdot \log_4 5^{3y} = 3x \log_5 4 \cdot 3y \log_4 5 = 9xy$$

(since $\log_5 4 \cdot \log_4 5 = 1$), so $xy = 5$. Substituting into the first equation: $\sqrt{x-4} + \sqrt{x-5} - 1 = 7x - x^2 - 10 = (x-5)(2-x)$. For $x \geq 5$ the left-hand side is strictly increasing, nonnegative, and 0 only at $x = 5$; the right-hand side is not positive, and 0 at $x = 5$. So $x = 5$, and the solution is $(5, 1)$.

b) $x = -1$ is clearly not a solution. For $x \neq -1$, the first equation gives $y = \frac{2}{x+1}$, and substituting into the second:

$$\left(1 - \frac{2}{x+1}\right) \cdot 2 \cos \frac{x\pi}{3} = \frac{x-1}{x+1}, \quad \frac{x-1}{x+1} \left(2 \cos \frac{x\pi}{3} - 1\right) = 0$$

So $x = 1$, or $\cos \frac{x\pi}{3} = \frac{1}{2}$, that is, $\frac{x\pi}{3} = \pm \frac{\pi}{3} + 2k\pi$: $x = 6k + 1$ or $x = 6l + 5$ (k, l integers); $x = 1$ is among these. The solutions are

$$\left(6k + 1, \frac{1}{3k + 1}\right), \quad k \in \mathbb{Z}, \quad \text{and} \quad \left(6l + 5, \frac{1}{3l + 3}\right), \quad l \in \mathbb{Z}, l \neq -1$$

(Note: for $l = -1$ we would get $x = -1$, which is excluded; the KoMaL solution does not mention this.)