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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

### Part I

1. a) A school library of 1600 volumes consists of Hungarian and English books.  $p\%$  of the books are in Hungarian.  $p\%$  of these books can also be found in the library in English translation. The number of books originally written in English is 64. Find the ratio of the Hungarian and English books.
- b) At the end of the year, during the renovation following the purchases, all the books are packed into identical boxes. If we put 25 books in each box, 17 books are left over; if we pack 27 books in each box, then four boxes remain empty, and the last box packed could still hold 5 more books. How many boxes did we have for the packing? By how many books did the library's collection grow by the end of the year?

Solution: a) There are  $1600 \cdot \frac{p}{100}$  Hungarian books, and  $1600 \cdot \frac{p^2}{10\,000}$  of them have been translated into English. Adding the books originally written in English, we get all the English books, which make up  $(100 - p)\%$  of the library:

$$1600 \cdot \frac{p^2}{10\,000} + 64 = 1600 \cdot \frac{100 - p}{100}, \quad p^2 + 100p - 9600 = 0$$

The only positive solution is  $p = 60$ : there are 960 Hungarian and 640 English books, so their ratio is 3 : 2.

- b) If there are  $d$  boxes, the number of books is  $25d + 17$  and also  $27(d - 5) + 22$ . From  $25d + 17 = 27d - 113$ ,  $d = 65$ . So there are 1642 books: the collection grew by 42 books.

2. We draw tangent lines to the circle  $x^2 + y^2 = 25$  at its points with  $x$ -coordinate 3.

- a) Write the equations of the tangent lines.
- b) Find the angle between the tangent lines.

Solution: a) The points of tangency are  $P_1(3, 4)$  and  $P_2(3, -4)$ , and the center of the circle is the origin, so normal vectors of the tangents are  $(3, 4)$  and  $(3, -4)$ . The tangents are  $3x + 4y = 25$  and  $3x - 4y = 25$ .

- b) The angle of the lines is equal to the angle of their normal vectors (or its supplement, if that is obtuse). Both normal vectors have length 5, and their dot product is  $9 - 16 = -7$ , so  $\cos \alpha = -\frac{7}{25}$  and  $\alpha \approx 106.26^\circ$ . This is obtuse, so the angle of the tangent lines is  $180^\circ - 106.26^\circ \approx 73.74^\circ$  ( $\approx 1.29$  radians).

3. Which real numbers satisfy the following equation?

$$\log_{4-5x}(4x^2 + 1 - 4x) + \log_{2x-1}(13x - 4 - 10x^2) = 4$$

Solution: Since  $4x^2 + 1 - 4x = (2x - 1)^2$  and  $13x - 4 - 10x^2 = (4 - 5x)(2x - 1)$ , the domain is given by  $4 - 5x > 0$ ,  $4 - 5x \neq 1$ ,  $2x - 1 > 0$ ,  $2x - 1 \neq 1$ , that is,  $\frac{1}{2} < x < \frac{4}{5}$  and  $x \neq \frac{3}{5}$ . Then the equation is

$$2 \log_{4-5x}(2x - 1) + \log_{2x-1}(4 - 5x) + 1 = 4$$

With  $y = \log_{4-5x}(2x - 1)$  (and  $\log_{2x-1}(4 - 5x) = \frac{1}{y}$ ) this is  $2y + \frac{1}{y} = 3$ , whose solutions are  $y = 1$  and  $y = \frac{1}{2}$ .

From  $y = 1$ :  $2x - 1 = 4 - 5x$ , so  $x = \frac{5}{7}$ , which is a solution.

From  $y = \frac{1}{2}$ :  $2x - 1 = \sqrt{4 - 5x}$ , which gives  $4x^2 + x - 5 = 0$  with the condition  $2x - 1 > 0$ , so  $x = \frac{3}{4}$  ( $x = -\frac{5}{4}$  is not a solution).

The solutions are  $\frac{3}{4}$  and  $\frac{5}{7}$ .

4. In a family of five, the sum of the ages of the parents is 80 years; the father is the older of them. The ages of the three sons are three consecutive terms of an arithmetic sequence with a prime difference, and their sum is 30. How old are the members of the family, if two years ago the father was three times as old as the oldest son, at whose birth the mother was older than 20?

Solution: Let the father be  $x$  years old; then the mother is  $80 - x$ , and since the father is older,  $x \geq 41$ . The ages of the sons are  $10 - p$ ,  $10$ ,  $10 + p$ , where  $p$  is a prime. Two years ago the father was  $x - 2$  and the oldest son  $8 + p$  years old, so  $x - 2 = 3(8 + p)$ , that is,  $x = 26 + 3p$ , and  $26 + 3p \geq 41$  gives  $p \geq 5$ . The difference of the ages of the mother and the oldest son is  $(80 - x) - (10 + p) > 20$ , so  $x < 50 - p$ , that is,  $26 + 3p < 50 - p$ , and  $p < 6$ . So  $p = 5$ : the father is 41, the mother 39, and the sons are 5, 10, and 15 years old.

## Part II

5. A box contains eight balls that are identical to the touch. The number 3 is written on one of them, 2 on two of them, and 1 on five of them. Someone takes out three balls and adds the numbers read from the balls.
- a) What results can be obtained this way?
- b) What is the probability of each result?

Solution: a) The possible sums are  $3 = 1 + 1 + 1$ ,  $4 = 1 + 1 + 2$ ,  $5 = 1 + 1 + 3 = 1 + 2 + 2$ ,  $6 = 1 + 2 + 3$ ,  $7 = 2 + 2 + 3$ .

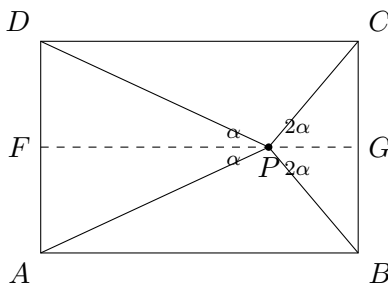
b) Three balls can be chosen in  $\binom{8}{3} = 56$  ways. The number of favorable cases and the probabilities:

$$P_3 = \frac{\binom{5}{3}}{56} = \frac{10}{56} \approx 0.1786, \quad P_4 = \frac{\binom{5}{2} \cdot 2}{56} = \frac{20}{56} \approx 0.3571, \quad P_5 = \frac{\binom{5}{2} \cdot 1 + 5 \cdot \binom{2}{2}}{56} = \frac{15}{56} \approx 0.2679,$$

$$P_6 = \frac{5 \cdot 2 \cdot 1}{56} = \frac{10}{56} \approx 0.1786, \quad P_7 = \frac{\binom{2}{2} \cdot 1}{56} = \frac{1}{56} \approx 0.0179$$

6. Two sides of a rectangle are 6 cm and 4 cm long. Where is the point  $P$  on the longer midline from which one of the 4 cm sides is seen at an angle twice as large as the other 4 cm side?

Solution: Let the rectangle be  $ABCD$  ( $AB = 6$ ,  $AD = 4$ ), and let the longer midline  $FG$  join the midpoints of  $AD$  and  $BC$ . The midline is an axis of symmetry, so the angles marked the same in the figure are equal. Let  $FP = x$ ; then  $PG = 6 - x$ .



In the right triangle  $AFP$ ,  $\tan \alpha = \frac{2}{x}$ , and in the right triangle  $BGP$ ,  $\tan 2\alpha = \frac{2}{6-x}$ . Substituting into the identity  $\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$ :

$$\frac{2}{6-x} = \frac{\frac{4}{x}}{1 - \frac{4}{x^2}}, \quad 3x^2 - 12x - 4 = 0, \quad x = 2 + \frac{4\sqrt{3}}{3} \approx 4.309$$

(the other root is negative). There are two such points on the longer midline, symmetric about the center of the rectangle, at distance  $\frac{4\sqrt{3}}{3} - 1 \approx 1.309$  cm from the center.

7. The base of a right pyramid with four lateral faces is a square of side  $a$ , and every lateral edge makes a  $30^\circ$  angle with the height of the pyramid. What is the area of the plane figure in which the pyramid is cut by the plane that passes through a vertex of the base of the pyramid and is perpendicular to the opposite lateral edge?

Solution: Let the pyramid be  $SABCD$  with base center  $O$ , and consider the plane through  $A$  perpendicular to  $SC$ . It meets  $SC$  at  $E$ ,  $SB$  at  $M$ ,  $SD$  at  $N$ , and the height  $SO$  at  $G$ . Since the lateral edges make  $30^\circ$  angles with the height,  $\angle ASO = \angle CSO = 30^\circ$ , so  $\angle ASC = 60^\circ$  and  $SA = SC$ : the triangle  $ACS$  is equilateral with side  $AC = a\sqrt{2}$ . So  $AE$ , its altitude, is  $AE = a\sqrt{2} \cdot \frac{\sqrt{3}}{2} = \frac{a\sqrt{6}}{2}$ .

By the symmetry of the pyramid,  $NM \parallel DB$ . In the equilateral triangle  $ACS$ ,  $G$  (the intersection of the altitudes  $AE$  and  $SO$ ) is also the centroid, so it divides  $SO$  in the ratio  $2 : 1$ ;  $G$  is also on  $NM$ , so  $NM$  is the segment through  $G$  parallel to  $DB$  in the triangle  $SBD$ , and by the intercept theorem  $MN = \frac{2}{3}DB = \frac{2}{3}a\sqrt{2}$ .

The cross section  $AMEN$  is a kite (symmetric about the plane  $ACS$ ), whose diagonals  $AE$  and  $MN$  are perpendicular. Its area is

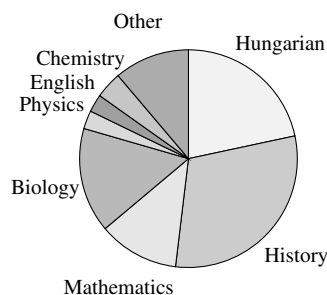
$$T = \frac{AE \cdot MN}{2} = \frac{\frac{a\sqrt{6}}{2} \cdot \frac{2a\sqrt{2}}{3}}{2} = \frac{a^2\sqrt{3}}{3}$$

8. The following table shows the distribution of the students' complaints submitted for all subjects about the grading of the advanced-level written exams of the May–June 2006 school-leaving exam.

| Subject                           | Number of complaints | Number of written exams |
|-----------------------------------|----------------------|-------------------------|
| Hungarian language and literature | 1187                 | 6251                    |
| History                           | 1660                 | 11 617                  |
| Mathematics                       | 657                  | 6408                    |
| Biology                           | 855                  | 5530                    |
| Physics                           | 149                  | 1745                    |
| English                           | 145                  | 3956                    |
| Chemistry                         | 218                  | 1814                    |
| Other                             | 614                  | 8700                    |
| Total                             | 5485                 | 46 021                  |

- Show the distribution of the complaints in a pie chart.
- What is the range of the number of complaints, and of the number of written exams, for the seven subjects?
- Find the standard deviation of the number of complaints, taking the seven subjects into account.
- In which subject did the students “complain” the most?
- Compute, for the seven subjects, the ratio of the number of complaints (compared to the number of written exams) in percent, and find the mean absolute deviation from the overall ratio.

Solution: a) The pie chart of the complaints:



- The range of the number of complaints is  $1660 - 145 = 1515$ , and of the number of written exams  $11\,617 - 1745 = 9872$ .
- The mean of the seven numbers of complaints is  $\bar{a} \approx 696$ , and the standard deviation is

$$\sqrt{\frac{(1187 - \bar{a})^2 + (1660 - \bar{a})^2 + \dots + (218 - \bar{a})^2}{7}} \approx 538$$

- In Hungarian language and literature, where the ratio is the largest (18.99%).
- The ratios are: Hungarian 18.99%, History 14.29%, Mathematics 10.25%, Biology 15.46%, Physics 8.54%, English 3.67%, Chemistry 12.02%. The overall ratio (all complaints compared to all written exams) is 11.92%, and the mean absolute deviation from it is

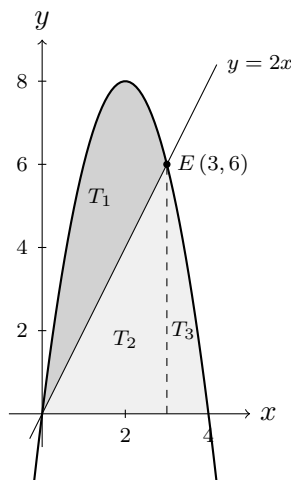
$$\frac{|18.99 - 11.92| + |14.29 - 11.92| + \dots + |12.02 - 11.92|}{7} \approx 3.77$$

9. The parabola  $y = ax^2 + bx + c$  passes through the origin, and the slope of its tangent line at its point  $E(3, 6)$  is  $-4$ .
- Write the equation of the parabola.
  - Give the equation of the tangent line mentioned above.
  - In what ratio does the line  $y = 2x$  divide the region bounded by the parabola and the  $x$ -axis?

Solution: a) The parabola passes through the origin, so  $c = 0$ ; it passes through  $E(3, 6)$ , so  $9a + 3b = 6$ . The slope of the tangent is  $y' = 2ax + b$ , and at  $x = 3$  it is  $6a + b = -4$ . So  $a = -2$ ,  $b = 8$ , and the parabola is  $y = -2x^2 + 8x$ .

b) The tangent passes through  $E(3, 6)$  with slope  $-4$ :  $y = -4x + 18$ .

c) The zeros of the parabola are 0 and 4, and the line  $y = 2x$  meets it at  $(0, 0)$  and  $E(3, 6)$ .



The part above the line is

$$T_1 = \int_0^3 (-2x^2 + 8x - 2x) dx = \left[ -\frac{2}{3}x^3 + 3x^2 \right]_0^3 = 9$$

The part below the line consists of the triangle  $T_2 = \frac{3 \cdot 6}{2} = 9$  and  $T_3 = \int_3^4 (-2x^2 + 8x) dx = \frac{10}{3}$ . The ratio is

$$\frac{T_1}{T_2 + T_3} = \frac{9}{9 + \frac{10}{3}} = \frac{27}{37}.$$