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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. Solve the following equations over the real numbers:

a) $(4 - x^2) \sqrt{1 - x} = 0$; b) $(0.4^x - 2.5^{x+1} - 1.5) \cdot \log_3(x + 3) = 0$; c) $\sin x = 5 \cos \frac{x}{2}$.

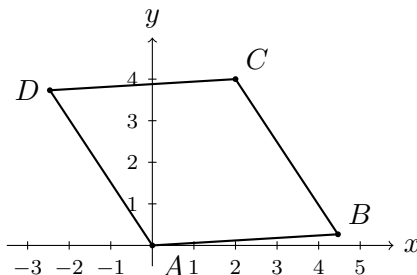
Solution: a) Because of the square root, $x \leq 1$. A product is 0 if one of its factors is 0: $4 - x^2 = 0$ gives $x = \pm 2$, but $x = 2$ is not in the domain; $\sqrt{1 - x} = 0$ gives $x = 1$. The solutions are $x = 1$ and $x = -2$.

b) Because of the logarithm, $x > -3$. With $t = \left(\frac{2}{5}\right)^x > 0$ we have $2.5^{x+1} = \frac{5}{2t}$, so $0.4^x - 2.5^{x+1} - 1.5 = 0$ means $t^2 - \frac{3}{2}t - \frac{5}{2} = 0$, that is, $t = \frac{5}{2}$ (or $t = -1$, which is impossible): $x = -1$. $\log_3(x + 3) = 0$ gives $x = -2$. The solutions are $x = -1$ and $x = -2$.

c) Since $\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}$, the equation is $\cos \frac{x}{2} \left(2 \sin \frac{x}{2} - 5\right) = 0$. $\sin \frac{x}{2} = \frac{5}{2}$ is impossible, so $\cos \frac{x}{2} = 0$, and $x = \pi + 2k\pi$ (k an integer).

2. Two opposite vertices of the rhombus $ABCD$ are $A(0, 0)$ and $C(2, 4)$. The angles at the vertices A and C are 120° . Find the coordinates of the vertices B and D .

Solution: The diagonals of a rhombus bisect its angles, so the triangles ABC and ACD are equilateral, with sides $AC = \sqrt{20}$. So the missing vertices are the intersections of the circles of radius $\sqrt{20}$ centered at A and at C , that is, the solutions of the system $x^2 + y^2 = 20$, $(x - 2)^2 + (y - 4)^2 = 20$:



The vertices are $B(1 + 2\sqrt{3}, 2 - \sqrt{3})$ and $D(1 - 2\sqrt{3}, 2 + \sqrt{3})$ (or the other way around).

3. Solve the following system of equations over the real numbers:

$$x^3 - x^2y - 4xy^2 + 4y^3 = 0, \quad 2x^2 + y^2 = 12$$

Solution: The first equation factors:

$$x^3 - x^2y - 4xy^2 + 4y^3 = x^2(x - y) - 4y^2(x - y) = (x - y)(x - 2y)(x + 2y)$$

So $x = y$, or $x = 2y$, or $x = -2y$. Substituting into the second equation, the solutions are

$$(2, 2), (-2, -2), \left(\frac{4\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}\right), \left(-\frac{4\sqrt{3}}{3}, -\frac{2\sqrt{3}}{3}\right), \left(-\frac{4\sqrt{3}}{3}, \frac{2\sqrt{3}}{3}\right), \left(\frac{4\sqrt{3}}{3}, -\frac{2\sqrt{3}}{3}\right)$$

4. The perimeter of the parallelogram $ABCD$ is 26 units, $AB < AD$, the radius of the circle inscribed in the triangle BCD is $\sqrt{3}$ units, and the angle of the parallelogram at the vertex B is 120° . Find the sides of the parallelogram.

Solution: Let $AB = CD = a$; then $BC = AD = 13 - a$, and the angle at C is 60° . Let the inscribed circle of the triangle BCD (center O) touch CD at P , BC at R , and BD at Q . In the right triangle OPC , $\angle POC = 60^\circ$ and $OP = \sqrt{3}$, so $PC = 3$. The tangent segments from a point are equal: $RC = PC = 3$, $DP = DQ = a - 3$, $BR = BQ = 13 - a - 3 = 10 - a$. So $DB = 7$. By the Law of Cosines in the triangle ABD :

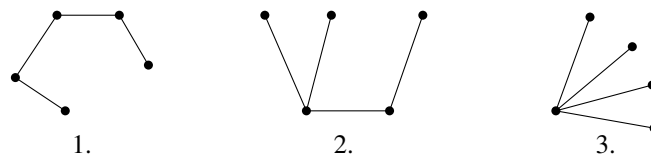
$$49 = a^2 + (13 - a)^2 - 2a(13 - a)\cos 60^\circ$$

so $a = 8$ or $a = 5$. Since $AB < AD$, the sides are $AB = CD = 5$ and $BC = AD = 8$ units.

Part II

5. Five points are given so that no three of them are on a line. We want to connect them with a network consisting of four straight segments; crossings are allowed. How many such networks are possible?

Solution: (The network must connect all five points, so it is a tree.) There are three types of networks:



Type 1 (a path): $\frac{5!}{2} = 60$ networks, since the starting point can be any of the five points, the next one any of the remaining four, and so on, and it does not matter which end is the start.

Type 2 (one point of degree 3): also 60, since the point of degree 3 can be chosen in 5 ways, the point of degree 2 in 4 ways, and the point joined only to the latter in 3 ways; then the other two points are determined.

Type 3 (a star): 5, since any point can be the center.

In all there are 125 possible networks. (This agrees with Cayley's formula $n^{n-2} = 5^3$ for the number of trees on n labeled points.)

6. Prove that in every triangle

$$\sin^2 \frac{\alpha - \beta}{2} + \sin \alpha \cdot \sin \beta + \sin^2 \frac{\gamma}{2} = 1$$

where α, β, γ are the angles of the triangle.

Solution: Since $\alpha + \beta + \gamma = 180^\circ$, $\frac{\gamma}{2} = 90^\circ - \frac{\alpha + \beta}{2}$, so $\sin^2 \frac{\gamma}{2} = \cos^2 \frac{\alpha + \beta}{2}$. With $s_\alpha = \sin \frac{\alpha}{2}$, $c_\alpha = \cos \frac{\alpha}{2}$, and similarly for β :

$$\begin{aligned} & \sin^2 \frac{\alpha - \beta}{2} + \sin \alpha \sin \beta + \cos^2 \frac{\alpha + \beta}{2} \\ &= (s_\alpha c_\beta - c_\alpha s_\beta)^2 + 4s_\alpha c_\alpha s_\beta c_\beta + (c_\alpha c_\beta - s_\alpha s_\beta)^2 \\ &= s_\alpha^2 (c_\beta^2 + s_\beta^2) + c_\alpha^2 (c_\beta^2 + s_\beta^2) = s_\alpha^2 + c_\alpha^2 = 1 \end{aligned}$$

7. A flea jumps on a long stick. Each jump is randomly to the left or to the right, and each jump is 10 cm long.

a) In how many ways can it get, with 10 jumps, 40 cm to the right of the starting point, and 50 cm to the left?

b) Find the probability that after 10 jumps the flea is at each of the points of the stick.

Solution: a) If the flea makes b jumps to the right and $10 - b$ to the left, it ends up $(b - (10 - b)) \cdot 10 = (2b - 10) \cdot 10$ cm to the right of the starting point. This distance is a multiple of 20 cm, so it cannot get 50 cm away. It gets 40 cm to the right if $2b - 10 = 4$, that is, $b = 7$: 7 jumps to the right and 3 to the left, which can be done in $\binom{10}{3} = 120$ ways.

b) The flea can make the 10 jumps in 2^{10} equally likely ways. It can only end up at the points $(2b - 10) \cdot 10$ cm to the right of the start ($b = 0, 1, \dots, 10$; a negative value means to the left), that is, at $-100, -80, \dots, 80, 100$ cm. It gets to the point $(2b - 10) \cdot 10$ cm in $\binom{10}{b}$ ways, so the probability of being there is

$$\frac{\binom{10}{b}}{2^{10}}, \quad b = 0, 1, \dots, 10$$

and at every other point the probability is 0.

8. In the function $f(x) = (p + 1)x^2 - (p + 3)x + 2p$, p is a real parameter. Find the values of p for which the function takes positive values for every real number x .

Solution: If $p = -1$, then f is a nonconstant linear function, which also takes negative values. If $p \neq -1$, then f is a quadratic function, which is positive for every real x exactly when $p + 1 > 0$ and its discriminant is negative:

$$D = (p + 3)^2 - 8p(p + 1) = -7p^2 - 2p + 9 = -(7p + 9)(p - 1) < 0$$

that is, $p < -\frac{9}{7}$ or $p > 1$. Together with $p > -1$: the function is positive everywhere exactly when $p > 1$.

9. Find the pairs (x, y) that satisfy the following equations:

a) $\sin^2(x + y) - \cos^2(x - y) = 1$; b) $\frac{1}{\sqrt{2}}(\sin x + \cos x) = 2y^2 - 4y + 3$.

Solution: a) Since $\sin^2(x + y) = 1 - \cos^2(x + y)$, the equation is $\cos^2(x + y) + \cos^2(x - y) = 0$. The sum of two squares is 0 only if both are 0: $x + y = \frac{\pi}{2} + n\pi$ and $x - y = \frac{\pi}{2} + k\pi$, so

$$x = \frac{\pi}{2} + (n + k) \frac{\pi}{2}, \quad y = (n - k) \frac{\pi}{2}, \quad n, k \text{ integers}$$

b) The left-hand side is $\sin\left(x + \frac{\pi}{4}\right) \leq 1$, and the right-hand side is $2(y - 1)^2 + 1 \geq 1$. So both are 1: $x = \frac{\pi}{4} + 2k\pi$ (k an integer) and $y = 1$; these pairs indeed satisfy the equation.