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translated from: KoMaL, January 2006, pages 16-17

Pdf file: KoMaL, January 2006, pages 16-17

From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. Find the value of k for which the following equation has a real root:

$$\sqrt{x^2 - 2005x - 2006} + \sqrt{x^2 - 2007x + 2006} + \sqrt{x^2 - k^2} = 0$$

(11 points)

2. Of three consecutive terms of a geometric sequence, the third is -2 . If we put it in front of the first two, we get three consecutive terms of an arithmetic sequence. Find the original three numbers.

(12 points)

3. Find the following sum to one decimal place:

$$\lg\left(7 \cdot \sqrt{\frac{1}{2}}\right) + \lg\left(7^2 \cdot \sqrt{\frac{2}{3}}\right) + \lg\left(7^3 \cdot \sqrt{\frac{3}{4}}\right) + \dots + \lg\left(7^{999} \cdot \sqrt{\frac{999}{1000}}\right)$$

(lg is the base 10 logarithm.)

(14 points)

4. The base AB of the isosceles triangle ABC is 56 cm, and its legs are 53 cm long. We know that the midpoint F of the base AB , the center O of the inscribed circle, the center K of the circumscribed circle, the centroid S , the orthocenter M , and the vertex C are on one line.

- How many triangles are determined by the points F , O , K , S , M , C , and A ?
- Compute the area of the triangle ASK .
- Compute the length FO .

(14 points)

Part II

5. If we rotate the graphs of the functions $f(x) = 3|x|$ and $g(x) = x^2$, defined on the interval $[-3, 3]$, around the y -axis, we get two cup-shaped solids. Find the difference of the volumes of the two cups in deciliters, if the unit of the coordinate system is 1.5 cm.

(16 points)

6. Solve the following equation over the real numbers:

$$\left(\frac{\sqrt{5}}{11}\right)^{\frac{2}{-5+\lg x} - \frac{4}{1+\lg x}} = 24.2$$

(16 points)

7. Three circles are given by their equations:

$$x^2 + 8x + y^2 - 4y + 16 = 0, \quad x^2 + 2x + y^2 - 6y + 6 = 0, \quad x^2 - 4x + y^2 + 12y + 36 = 0$$

- a) Compute the perimeter and the area of the triangle determined by their centers.
- b) Write the equation of the circle that the given circles touch from inside.

(16 points)

8. András and Béla play a lot of table tennis with each other. Experience shows that András wins a game with probability 0.7, and Béla with probability 0.3. If they play several times, the winner is the one who wins more games.

- a) What is András's probability of winning, if they play four times?
- b) What is the probability that Béla wins every game, if they play three times?
- c) How did András's probability of winning one game change, if in two games Béla does not lose with probability $\frac{5}{9}$?

(16 points)

9. Prove that the inequality $2^n \cdot n! < (n+1)^n$ holds for every natural number n greater than 1.

(16 points)