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translated from: KoMaL, January 2006, pages 16-17

Pdf file: KoMaL, January 2006, pages 16-17

From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. Find the value of k for which the following equation has a real root:

$$\sqrt{x^2 - 2005x - 2006} + \sqrt{x^2 - 2007x + 2006} + \sqrt{x^2 - k^2} = 0$$

(11 points)

Solution: The sum of three nonnegative real numbers can be 0 only if each of them is 0. The quadratic expressions factor:

$$\sqrt{(x+1)(x-2006)} + \sqrt{(x-1)(x-2006)} + \sqrt{(x+k)(x-k)} = 0$$

The first two terms are both 0 only for $x = 2006$. So if there is a solution, the third term must also be 0 at $x = 2006$: $2006^2 - k^2 = 0$, that is, $k = 2006$ or $k = -2006$.

2. Of three consecutive terms of a geometric sequence, the third is -2 . If we put it in front of the first two, we get three consecutive terms of an arithmetic sequence. Find the original three numbers.

(12 points)

Solution: Let the terms of the arithmetic sequence be $-2, -2 + d, -2 + 2d$; then the terms of the geometric sequence are $-2 + d, -2 + 2d, -2$, so

$$(-2 + 2d)^2 = -2(-2 + d), \quad 4d^2 - 6d = 0$$

so $d = 0$ or $d = 1.5$. The original three numbers are $-2, -2, -2$, or $-0.5, 1, -2$; both triples satisfy the conditions.

3. Find the following sum to one decimal place:

$$\lg\left(7 \cdot \sqrt{\frac{1}{2}}\right) + \lg\left(7^2 \cdot \sqrt{\frac{2}{3}}\right) + \lg\left(7^3 \cdot \sqrt{\frac{3}{4}}\right) + \dots + \lg\left(7^{999} \cdot \sqrt{\frac{999}{1000}}\right)$$

(lg is the base 10 logarithm.)

(14 points)

Solution: Using the identities of logarithms, the sum is

$$(\lg 7 + \lg 7^2 + \dots + \lg 7^{999}) + \left(\lg \sqrt{\frac{1}{2}} + \lg \sqrt{\frac{2}{3}} + \dots + \lg \sqrt{\frac{999}{1000}}\right)$$

The first part is $(1 + 2 + \dots + 999) \lg 7 = \frac{999 \cdot 1000}{2} \lg 7 = 499\,500 \lg 7$. The second part is

$$\lg \sqrt{\frac{1}{2} \cdot \frac{2}{3} \cdot \dots \cdot \frac{999}{1000}} = \lg \sqrt{\frac{1}{1000}} = -\frac{3}{2}$$

So the sum is $499\,500 \lg 7 - \frac{3}{2} \approx 422\,125.0$.

4. The base AB of the isosceles triangle ABC is 56 cm, and its legs are 53 cm long. We know that the midpoint F of the base AB , the center O of the inscribed circle, the center K of the circumscribed circle, the centroid S , the orthocenter M , and the vertex C are on one line.

- How many triangles are determined by the points F, O, K, S, M, C , and A ?
- Compute the area of the triangle ASK .
- Compute the length FO .

(14 points)

Solution: a) The points F, O, K, S, M, C are on one line, so a (nondegenerate) triangle has the vertex A and two of these six points: $\binom{6}{2} = 15$ triangles.

b) The altitude from A in the triangle ASK (to the line CF) is $AF = 28$ cm. By the Pythagorean Theorem in the triangle ACF , $28^2 + m_c^2 = 53^2$, so $m_c = CF = 45$ cm. The center K of the circumscribed circle is $45 - R$ from the base, where $28^2 + (45 - R)^2 = R^2$, so $R = \frac{2809}{90}$ cm, and $FK = 45 - \frac{2809}{90} = \frac{1241}{90} \approx 13.79$ cm. The centroid is $\frac{45}{3} = 15$ cm from the base. So $SK = 15 - \frac{1241}{90} = \frac{109}{90}$ cm, and

$$T_{ASK} = \frac{28 \cdot \frac{109}{90}}{2} = \frac{1526}{90} \approx 16.96 \text{ cm}^2$$

c) The center of the inscribed circle is r from the base. The area of the triangle is $t = \frac{56 \cdot 45}{2} = 1260 \text{ cm}^2$, and its semiperimeter is $s = 81$ cm; from $t = rs$, $r = \frac{140}{9}$ cm. So $FO = \frac{140}{9} \approx 15.56$ cm.

Part II

5. If we rotate the graphs of the functions $f(x) = 3|x|$ and $g(x) = x^2$, defined on the interval $[-3, 3]$, around the y -axis, we get two cup-shaped solids. Find the difference of the volumes of the two cups in deciliters, if the unit of the coordinate system is 1.5 cm.

(16 points)

Solution: The first cup is a cone with base radius 3 and height 9; its volume is $V_1 = \frac{9\pi \cdot 9}{3} = 27\pi \approx 84.82$ cubic units. One unit of length is 1.5 cm, so we multiply the volumes by 1.5^3 : about $286.3 \text{ cm}^3 = 2.863 \text{ dl}$.

The second cup is the same solid we get by rotating the graph of $h(x) = \sqrt{x}$ on $[0, 9]$ around the x -axis, so its volume is

$$V_2 = \pi \int_0^9 (\sqrt{x})^2 dx = \pi \left[\frac{x^2}{2} \right]_0^9 = \frac{81\pi}{2} \approx 127.23 \text{ cubic units}$$

which is about $429.4 \text{ cm}^3 = 4.294 \text{ dl}$. So the second cup holds about 1.431 dl more than the first.

6. Solve the following equation over the real numbers:

$$\left(\frac{\sqrt{5}}{11}\right)^{\frac{2}{-5+\lg x} - \frac{4}{1+\lg x}} = 24.2$$

(16 points)

Solution: We need $x > 0$, and the denominators cannot be 0, so $x \neq 10^5$ and $x \neq 10^{-1}$. Since

$$24.2 = \frac{121}{5} = \left(\frac{11}{\sqrt{5}}\right)^2 = \left(\frac{\sqrt{5}}{11}\right)^{-2}$$

the equation is equivalent to $\frac{2}{-5+\lg x} - \frac{4}{1+\lg x} = -2$. This can be brought to the form $\lg^2 x - 5\lg x + 6 = 0$, so $\lg x = 2$ or $\lg x = 3$. The solutions are $x = 100$ and $x = 1000$, and both are indeed solutions.

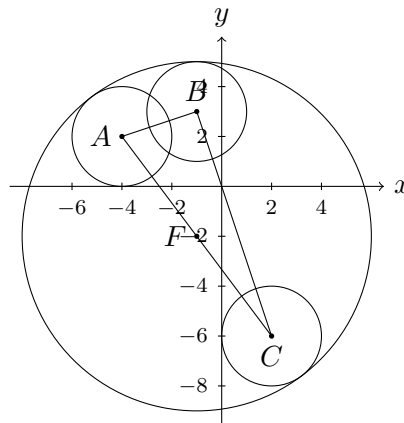
7. Three circles are given by their equations:

$$x^2 + 8x + y^2 - 4y + 16 = 0, \quad x^2 + 2x + y^2 - 6y + 6 = 0, \quad x^2 - 4x + y^2 + 12y + 36 = 0$$

- Compute the perimeter and the area of the triangle determined by their centers.
- Write the equation of the circle that the given circles touch from inside.

(16 points)

Solution: The circles are $(x+4)^2 + (y-2)^2 = 4$, $(x+1)^2 + (y-3)^2 = 4$, and $(x-2)^2 + (y+6)^2 = 4$, with centers $A(-4, 2)$, $B(-1, 3)$, $C(2, -6)$, and radius 2 each.



- $AC = 10$, $BC = 3\sqrt{10}$, $AB = \sqrt{10}$, so the perimeter is $10 + 4\sqrt{10} \approx 22.65$. Since $AC^2 = AB^2 + BC^2$, the triangle has a right angle at B , and its area is $\frac{3\sqrt{10} \cdot \sqrt{10}}{2} = 15$.
- The center of the circle is the midpoint $F(-1, -2)$ of the hypotenuse AC (the center of the circle circumscribed around ABC), and since each given circle has radius 2, its radius is 2 more than $FA = 5$, that is, 7. The circle is $(x+1)^2 + (y+2)^2 = 49$.

8. András and Béla play a lot of table tennis with each other. Experience shows that András wins a game with probability 0.7, and Béla with probability 0.3. If they play several times, the winner is the one who wins more games.
- What is András's probability of winning, if they play four times?
 - What is the probability that Béla wins every game, if they play three times?
 - How did András's probability of winning one game change, if in two games Béla does not lose with probability $\frac{5}{9}$?
- (16 points)

Solution: a) András wins if he wins three or four of the four games:

$$\binom{4}{3} \cdot 0.7^3 \cdot 0.3 + \binom{4}{4} \cdot 0.7^4 = 0.4116 + 0.2401 = 0.6517$$

b) $0.3^3 = 0.027$.

c) If András wins a game with probability p , then Béla does not lose in two games if András wins at most one of them:

$$(1 - p)^2 + 2p(1 - p) = 1 - p^2 = \frac{5}{9}$$

so $p^2 = \frac{4}{9}$ and $p = \frac{2}{3}$. András's probability of winning a game decreased from 0.7 to about 0.67.

9. Prove that the inequality $2^n \cdot n! < (n + 1)^n$ holds for every natural number n greater than 1.

(16 points)

Solution: Both sides are positive, so we can take their n th roots: the inequality is equivalent to $\sqrt[n]{1 \cdot 2 \cdot \dots \cdot n} < \frac{n + 1}{2}$. Since

$$\frac{n + 1}{2} = \frac{\frac{n(n+1)}{2}}{n} = \frac{1 + 2 + \dots + n}{n}$$

this is the inequality between the geometric and the arithmetic means of the numbers $1, 2, \dots, n$, which is strict, since for $n > 1$ these numbers are not all equal.