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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. a) Does the following equation have a solution over the positive integers?

$$1! + 2! + 3! + \dots + x! = 2y$$

- b) Solve the following equation:

$$\tan \alpha + \sqrt{3} = \frac{\sqrt{2}}{\cos \alpha}$$

Solution: a) x and y can only be positive integers. $1! = 1$, and for $n \geq 2$, $n!$ is even, since it contains the factor 2. So the left-hand side is odd, while the right-hand side is even: the equation has no solution.

b) We need $\cos \alpha \neq 0$. Multiplying both sides by $\cos \alpha$ and dividing by 2: $\frac{1}{2} \sin \alpha + \frac{\sqrt{3}}{2} \cos \alpha = \frac{\sqrt{2}}{2}$. Using $\cos \frac{\pi}{3} = \frac{1}{2}$ and $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and the addition formula, this is $\sin \left(\alpha + \frac{\pi}{3} \right) = \frac{\sqrt{2}}{2}$. The solutions are

$$\alpha = -\frac{\pi}{12} + 2k\pi \quad \text{and} \quad \alpha = \frac{5\pi}{12} + 2k\pi, \quad k \text{ an integer}$$

2. The table shows Sanyi's midyear report card (grades from 1 to 5, 5 is the best).

Hungarian literature	5 (excellent)
Hungarian language	4 (good)
History	4 (good)
Mathematics	3 (satisfactory)
English	4 (good)
German	3 (satisfactory)
Physics	2 (pass)
Chemistry	3 (satisfactory)
Biology	2 (pass)
Art	4 (good)
Music	5 (excellent)
Physical education	5 (excellent)

- Give the mode, the median, and the range of Sanyi's grades.
- Compute Sanyi's grade point average (the mean of the grades).
- Give the standard deviation of the grades.
- Show the percentage distribution of the grades in a pie chart.

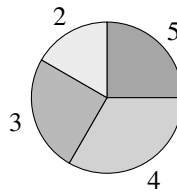
Solution: There are three 5s, four 4s, three 3s, and two 2s.

a) The mode is 4, the median is 4, and the range is $5 - 2 = 3$.

b) The mean is $\frac{44}{12} \approx 3.667$.

c) The standard deviation is $\sqrt{\frac{3\left(\frac{4}{3}\right)^2 + 4\left(\frac{1}{3}\right)^2 + 3\left(\frac{2}{3}\right)^2 + 2\left(\frac{5}{3}\right)^2}{12}} = \sqrt{\frac{38}{36}} \approx 1.027$.

d) 25% of the grades are 5s, 33.33% are 4s, 25% are 3s, and 16.67% are 2s:



- Find the positive integers x for which $x^4 + 4$ is a prime.

Solution: $x^4 + 4 = x^4 + 4x^2 + 4 - 4x^2 = (x^2 + 2)^2 - (2x)^2 = (x^2 - 2x + 2)(x^2 + 2x + 2)$. Since x is a positive integer, the factors are integers and $0 < x^2 - 2x + 2 < x^2 + 2x + 2$. The product can be a prime only if the smaller factor is 1: $x^2 - 2x + 2 = 1$, that is, $x = 1$. Then $x^4 + 4 = 5$, which is indeed a prime. So $x = 1$ is the only such number.

- The sum of the first and third terms of a geometric sequence is 40. If we add 4 to the first term and 6 to the second, then together with the third term we get three consecutive terms of an arithmetic sequence (in the same order). Which is this geometric sequence?

Solution: Let the terms of the geometric sequence be a_1, a_1q, a_1q^2 ; then $a_1 + 4, a_1q + 6, a_1q^2$ are consecutive terms of an arithmetic sequence. So

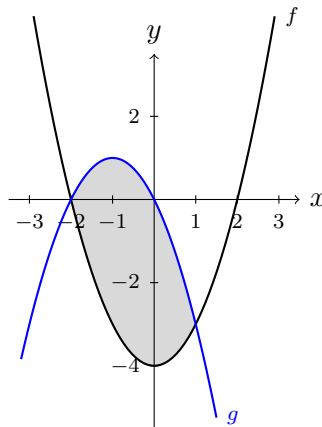
$$a_1 + a_1q^2 = 40 \quad \text{and} \quad (a_1q + 6) - (a_1 + 4) = a_1q^2 - (a_1q + 6)$$

From the second, $a_1(q - 1)^2 = 8$; from the first, $a_1(1 + q^2) = 40$. Dividing, $\frac{1 + q^2}{(q - 1)^2} = 5$, so $2q^2 - 5q + 2 = 0$, and $q = 2, a_1 = 8$, or $q = 0.5, a_1 = 32$. Both are solutions: 8, 16, 32, ... (with 12, 22, 32) and 32, 16, 8, ... (with 36, 22, 8).

Part II

5. In what ratio does the y -axis divide the region bounded by the graphs of the functions $f(x) = x^2 - 4$ and $g(x) = -x^2 - 2x$?

Solution: The graphs meet where $x^2 - 4 = -x^2 - 2x$, that is, at $x = -2$ and $x = 1$.



On the interval $(-2, 1)$ the difference $f(x) - g(x) = 2x^2 + 2x - 4$ has constant sign (it is negative), and

$$\int (2x^2 + 2x - 4) dx = \frac{2}{3}x^3 + x^2 - 4x + c$$

So

$$\left[\frac{2}{3}x^3 + x^2 - 4x \right]_{-2}^0 = -\frac{20}{3} \quad \text{and} \quad \left[\frac{2}{3}x^3 + x^2 - 4x \right]_0^1 = -\frac{7}{3}$$

The parts of the region to the left and to the right of the y -axis have areas $\frac{20}{3}$ and $\frac{7}{3}$, so the ratio is 20 : 7.

6. The base circle of a horizontal cylinder-shaped tank has radius 1 m, and the length of the cylinder is 3 m. The water in the tank is now 1.6 m deep. We let some of the water out into a vertical cylinder-shaped container that is large enough, with base radius 0.5 m. How high will the water be in this container, if the water level in the horizontal cylinder dropped by 0.3 m?

Solution: First, the cross section of the water in the horizontal cylinder: the empty part above the water is a circular segment whose chord is 0.6 m from the center. With $\cos \varphi = 0.6$, $\varphi \approx 53.13^\circ$, and the area of this segment is

$$\frac{2\varphi}{360^\circ} r^2 \pi - \frac{1}{2} r^2 \sin 2\varphi \approx 0.9273 - 0.4800 = 0.4473 \text{ m}^2$$

so the water's cross section is $T_1 \approx \pi - 0.4473 \approx 2.6943 \text{ m}^2$, and the volume of water is $V_1 = 3T_1 \approx 8.083 \text{ m}^3$.

After the level drops by 0.3 m, the water is 1.3 m deep, and the chord of the empty segment is 0.3 m from the center. With $\cos \varepsilon = 0.3$, $\varepsilon \approx 72.54^\circ$, and the segment's area is $\approx 1.2661 - 0.2862 = 0.9799 \text{ m}^2$, so $T_2 \approx 2.1617 \text{ m}^2$ and $V_2 \approx 6.485 \text{ m}^3$.

The water let out is $V \approx 1.598 \text{ m}^3$. In the vertical cylinder $V = 0.5^2 \pi m$, so $m \approx 2.03 \text{ m}$: the water will be about 2 m high.

7. A factory received an order to make 100 cone-shaped ice cream wafers, open at the top, each with a volume of 1 dm^3 . The material is expensive, and the customer only prescribed the volume of the cone. For what vertex angle (the angle of the axial cross section at the apex) of the cone is the material cost minimal?

Solution: The volume of the cone is $\frac{r^2\pi m}{3} = 1000 \text{ cm}^3$, so $m = \frac{3000}{r^2\pi}$. The slant height is $a = \sqrt{r^2 + m^2}$, and the area of the lateral surface is

$$A = r\pi a = r\pi\sqrt{r^2 + \left(\frac{3000}{r^2\pi}\right)^2} = \pi\sqrt{r^4 + \frac{9\,000\,000}{r^2\pi^2}}$$

It is enough to find the minimum of $h(r) = r^4 + \frac{9\,000\,000}{\pi^2 r^2}$ ($r > 0$). Setting the derivative $h'(r) = 4r^3 - \frac{18\,000\,000}{\pi^2 r^3}$ equal to 0:

$$r^6 = \frac{18\,000\,000}{4\pi^2} \approx 455\,945.33, \quad r \approx 8.773 \text{ cm}$$

h' changes from negative to positive here (or: $h''(r) = 12r^2 + \frac{54\,000\,000}{\pi^2 r^4} > 0$), so this is a minimum. Then $m \approx 12.407 \text{ cm}$ (in fact $m = \sqrt{2}r$), and $\tan \frac{\varphi}{2} = \frac{r}{m} = \frac{1}{\sqrt{2}}$, so the vertex angle is $\varphi \approx 70.53^\circ$.

8. Where can the right-angle vertex of the right triangles be whose area is 25 units and whose hypotenuse has endpoints $A(1, -2)$ and $B(6, 8)$?

Solution: The third vertex is on the Thales circle of the segment AB , and on one of the two lines e_1, e_2 parallel to AB at the distance m of the altitude. Since $AB = \sqrt{125} = 5\sqrt{5}$ and $T = \frac{AB \cdot m}{2} = 25$, $m = 2\sqrt{5}$. The Thales circle is $(x - 3.5)^2 + (y - 3)^2 = \frac{125}{4}$, and the line AB is $-2x + y = -4$, so the parallel lines at distance $2\sqrt{5}$ are

$$e_1: -2x + y = 6, \quad e_2: -2x + y = -14$$

(the radius of the circle, $\frac{5\sqrt{5}}{2}$, is greater than $2\sqrt{5}$, so both lines meet the circle). There are four solutions: the intersections of e_1 with the circle, $C_1(1, 8)$ and $C_2(-2, 2)$, and of e_2 with the circle, $C_3(9, 4)$ and $C_4(6, -2)$.

9. a) Are there, in a group of 25 people, 3 people who were born in the same month? Justify your answer.
 b) We choose an arbitrary subset of the set of one-digit positive integers. What is the probability that 1 or 2 is in it?
 c) At most how many subsets can we choose from the set of one-digit positive integers so that any two of them have a common element?

Solution: a) Yes. By the pigeonhole principle there is a month in which at least three of them were born, since $2 \cdot 12 = 24 < 25$. Any three of the people born in that month are suitable.

b) The set $\{1, 2, \dots, 9\}$ has 2^9 subsets, and 2^7 of them contain neither 1 nor 2. The probability is $1 - \frac{2^7}{2^9} = \frac{3}{4}$.

c) 2^8 . For example, the 2^8 subsets that contain 1 have this property. More cannot be chosen: the 2^9 subsets can be arranged into 2^8 pairs of complementary subsets, and two complementary subsets have no common element, so from each pair at most one can be chosen.