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Pdf file: KoMaL, November 2006, page 470

From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. According to Aladár, choosing among the three-digit numbers, and according to Barnabás, choosing among the five-digit numbers, the probability that the chosen number contains the digit 6 is greater. Which of them is right?

(11 points)

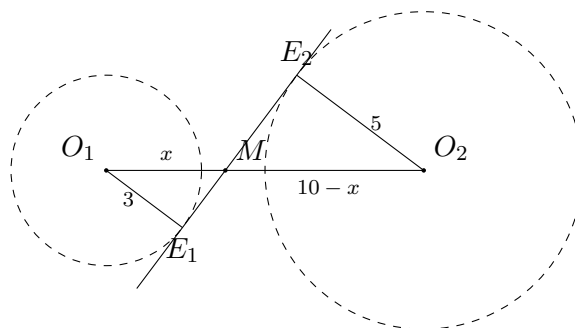
Solution: It is easier to count the numbers that do not contain the digit 6, and subtract their number from the number of all cases. Among the five-digit numbers, the first digit cannot be 0 or 6, and the other digits can be anything except 6: $8 \cdot 9 \cdot 9 \cdot 9 \cdot 9 = 52\,488$ numbers. So $90\,000 - 52\,488 = 37\,512$ five-digit numbers contain a 6, and the probability is $\frac{37\,512}{90\,000} \approx 0.417$.

Among the three-digit numbers, $8 \cdot 9 \cdot 9 = 648$ do not contain a 6, so $900 - 648 = 252$ do, and the probability is $\frac{252}{900} = 0.28$. So Barnabás is right: the probability is greater among the five-digit numbers.

2. Two circles with radii 5 cm and 3 cm are given. The centers of the circles are 10 cm apart. How far from the center of the smaller circle is the intersection point of the two common internal tangents of the circles?

(12 points)

Solution: The internal tangents meet on the segment O_1O_2 joining the centers, at a point M ; let O_1 be the center of the smaller circle and $O_1M = x$. Let an internal tangent touch the circles at E_1 and E_2 .



The right triangles O_1E_1M and O_2E_2M are similar (their angles at M are vertical angles), so $\frac{x}{3} = \frac{10 - x}{5}$, and $x = 3.75$ cm. The internal tangents meet 3.75 cm from the center of the smaller circle.

3. What is the remainder of $16^{101} + 8^{101} + 4^{101} + 2^{101} + 1$ when divided by $2^{100} + 1$?

(14 points)

Solution: We use that $a + b \mid a^{2k} - b^{2k}$ and $a + b \mid a^{2k+1} + b^{2k+1}$. With $a = 2^{100}$, $b = 1$:

$$\begin{aligned} 2^{100} + 1 &\mid 2^{400} - 1, \text{ so } 2^{100} + 1 \mid 16^{101} - 16, \\ 2^{100} + 1 &\mid 2^{300} + 1, \text{ so } 2^{100} + 1 \mid 8^{101} + 8, \\ 2^{100} + 1 &\mid 2^{200} - 1, \text{ so } 2^{100} + 1 \mid 4^{101} - 4, \\ 2^{100} + 1 &\mid 2^{100} + 1, \text{ so } 2^{100} + 1 \mid 2^{101} + 2. \end{aligned}$$

(For example, $16^{101} - 16 = 16(2^{400} - 1)$.) So

$$16^{101} + 8^{101} + 4^{101} + 2^{101} + 1 = (16^{101} - 16) + (8^{101} + 8) + (4^{101} - 4) + (2^{101} + 2) + 11$$

and the remainder is 11.

4. a) Find the linear functions $f(x)$ for which $2f(x) + 3f(1-x) = 4x - 1$.
b) What can the sign of c be, if the expression $ax^2 - bx + c$ is positive for every x ?

(14 points)

Solution: a) Substituting $1-x$ for x : $2f(1-x) + 3f(x) = 3 - 4x$. From the system

$$2f(x) + 3f(1-x) = 4x - 1, \quad 3f(x) + 2f(1-x) = 3 - 4x$$

we get $5f(x) = 11 - 20x$, so $f(x) = -4x + \frac{11}{5}$, and this function indeed satisfies the condition.

b) The value of the expression at $x = 0$ is c , so $c > 0$ is necessary: c must be positive.

Part II

5. Give all solutions of the following trigonometric equation in the interval $[0, 2\pi]$:

$$3 \tan\left(x + \frac{\pi}{3}\right) + \tan\left(x - \frac{\pi}{6}\right) = 0$$

(16 points)

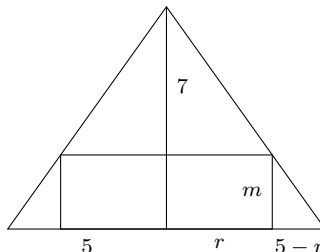
Solution: The domain is $x \neq \frac{\pi}{6} + k\pi$ and $x \neq \frac{2\pi}{3} + k\pi$ (k an integer). Notice that $x + \frac{\pi}{3} = \left(x - \frac{\pi}{6}\right) + \frac{\pi}{2}$. With $y = x - \frac{\pi}{6}$ the equation is $3 \tan\left(y + \frac{\pi}{2}\right) + \tan y = 0$, and since $\tan\left(y + \frac{\pi}{2}\right) = -\cot y = -\frac{1}{\tan y}$, it becomes $-\frac{3}{\tan y} + \tan y = 0$, that is, $\tan^2 y = 3$. So $y = \pm \frac{\pi}{3} + k\pi$, and $x = y + \frac{\pi}{6}$. In the interval $[0, 2\pi]$ the solutions are

$$x = \frac{\pi}{2}, \quad \frac{5\pi}{6}, \quad \frac{3\pi}{2}, \quad \frac{11\pi}{6}$$

6. What is the largest volume of a right circular cylinder that can be inscribed in a right circular cone whose base has radius 5 units and whose height is 7 units? (The cylinder and the cone have a common axis.)

(16 points)

Solution: Let the radius of the base of the cylinder be r ($0 < r < 5$) and its height m . In the axial cross section, by similar triangles $\frac{5-r}{m} = \frac{5}{7}$, so $m = 7 - \frac{7}{5}r$:



The volume of the cylinder is

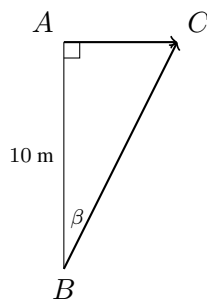
$$V(r) = r^2\pi \left(7 - \frac{7}{5}r\right) = -\frac{7\pi}{5}r^3 + 7\pi r^2, \quad V'(r) = -\frac{21\pi}{5}r^2 + 14\pi r$$

In $(0, 5)$ the derivative has a single zero, $r = \frac{10}{3}$, where it changes from positive to negative, so V has its maximum there. The cylinder of largest volume has height $\frac{7}{3}$ and volume $\frac{700\pi}{27} \approx 81.45$ cubic units.

7. A car starts and moves on a straight road with acceleration 2 m/s^2 . A man runs toward the car across the field from a road perpendicular to this road, at a constant speed of 5 m/s . How should he choose his starting direction so that he can jump into the car, if at the moment of starting he is 10 m from the car? After how much time does he reach the car?

(16 points)

Solution: The car starts at A and the man at B , with $AB = 10 \text{ m}$ perpendicular to the road; let C be the point where the man jumps into the car, t seconds after the start.



The man covers $BC = 5t$, and the car covers $AC = \frac{a}{2}t^2 = t^2$ (meters). In the right triangle ABC , with $\beta = \angle ABC$:

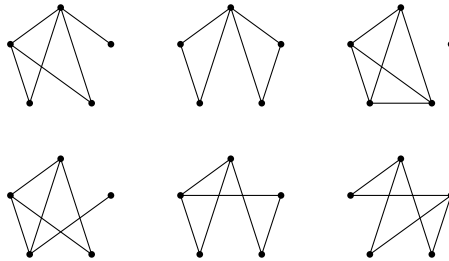
$$\sin \beta = \frac{AC}{BC} = \frac{t^2}{5t} = \frac{t}{5}, \quad \cos \beta = \frac{AB}{BC} = \frac{10}{5t} = \frac{2}{t}$$

Multiplying, $\sin \beta \cos \beta = \frac{2}{5}$, that is, $\sin 2\beta = \frac{4}{5}$, so $\beta \approx 26.57^\circ$ or $\beta \approx 63.43^\circ$ (only acute angles make sense). The man can choose two directions: if he runs at 26.57° to the road AB , he can jump in after $\sqrt{5} \approx 2.24 \text{ s}$; if at 63.43° , he reaches the car after $\sqrt{20} \approx 4.47 \text{ s}$. (Equivalently, $(5t)^2 = 100 + t^4$ gives $t^2 = 5$ or $t^2 = 20$.)

8. a) How many simple graphs with five vertices and six edges are there, if we distinguish the vertices, and how many, if we do not?
- b) János drew all graphs with five vertices and six edges, distinguishing the vertices. Lackó happened to pass by and pointed at a vertex. What is the probability that this vertex had degree 1?

(16 points)

Solution: a) The complete graph on 5 vertices has 10 edges, and 6 of them can be chosen in $\binom{10}{6} = 210$ ways: there are 210 such graphs if we distinguish the vertices. If we do not distinguish the vertices, there are 6 different graphs:



- b) Two of these graphs have a vertex of degree 1 (one each), and János drew 60 copies of each, so there are 120 vertices of degree 1 among the $210 \cdot 5 = 1050$ vertices. The probability is $\frac{120}{1050} = \frac{4}{35}$. (We checked this count by computer.)

9. Solve the following equation over the real numbers:

$$\sqrt[3]{2x^2 - 8x} - \sqrt[3]{2x^2} = 2$$

(16 points)

Solution: If $x \geq 0$, then $2x^2 - 8x \leq 2x^2$, so the left-hand side is at most 0, and there is no solution. Let $x < 0$, and let $b = \sqrt[3]{2x^2} > 0$, $a = \sqrt[3]{2x^2 - 8x}$. If $a = b + 2$, then

$$-8x = a^3 - b^3 = (b + 2)^3 - b^3 = 6b^2 + 12b + 8$$

Since $b^3 = 2x^2$ and $x < 0$, $-8x = 8|x| = 8\sqrt{\frac{b^3}{2}} = 4\sqrt{2}b^{\frac{3}{2}}$. With $s = \sqrt{b} > 0$ we get $6s^4 - 4\sqrt{2}s^3 + 12s^2 + 8 = 0$. But $6s^4 - 4\sqrt{2}s^3 + 12s^2 = s^2(6s^2 - 4\sqrt{2}s + 12) \geq 0$ (the quadratic factor has negative discriminant), so the left-hand side is at least 8. So the equation has no real solution.

(The KoMaL solution reaches the same conclusion using the graph of $t \mapsto t^{\frac{2}{3}}$; this algebraic argument is ours.)