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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. In the base 10 number $a = \overline{374x25y}$, the digits x and y are chosen at random. What is the probability that the number is divisible by 15?
2. Two circles with different radii touch each other externally. Prove that the segment of the common internal tangent between the two common external tangents is twice the geometric mean of the two radii.
3. Graph, in the Cartesian coordinate system, the sets of points that satisfy the following equations:
a) $(x - 3) \cdot (4x - 3y) = 0$; b) $17x^2 - 24xy + 9y^2 - 6x + 9 = 0$; c) $\frac{4x - 3y}{x - 3} = 0$; d) $\frac{x - 3}{4x - 3y} = 0$.
4. Solve the following inequality over the real numbers:

$$2^x + 2^{-x} + \frac{1}{2^x + 2^{-x}} \geq -\frac{11}{3\sqrt{2}} \log_{(3-2\sqrt{2})} (3 + 2\sqrt{2})$$

Part II

5. The harmonic mean of three positive numbers is $\frac{180}{13}$, their geometric mean is $10\sqrt[3]{3}$, and their quadratic mean is $\sqrt{\frac{725}{3}}$. What is their arithmetic mean?
6. a) In the quadratic equation $4px^2 - (2p^3 - 8p + 2)x + p^2 - 4 = 0$, find the values of the real parameter p for which the equation has real roots of opposite signs, neither of which is greater than $\frac{1}{2}$.
b) Is there a value of the parameter p for which the function $y = 4px^2 - (2p^3 - 8p + 2)x + p^2 - 4$ takes its maximum at $x = 2$, and this maximum is -8 ?
7. We rotate the points $A(a_1, a_2)$, $B(b_1, b_2)$, $C(-a_1, -a_2)$, and $D(-b_1, -b_2)$ around the origin by 90° in the positive direction, and then apply to the resulting points the dilation with center at the origin and ratio 2. Let the points obtained be A_1, B_1, C_1 , and D_1 . Prove that the midpoints of the segments AA_1, BB_1, CC_1 , and DD_1 determine a parallelogram, or lie on one line.
8. How many real solutions does the equation

$$\frac{x^2 - 4x + 4}{x^2 + 2x + 3} = k$$

have, depending on the real parameter k ?

9. A box contains n white and twice as many black balls ($n \geq 3$), which cannot be told apart by touch. We draw 3 balls without replacement.
- a) Let the random variable ξ be the number of white balls in the sample. Find the probabilities $P(\xi = 0)$, $P(\xi = 1)$, $P(\xi = 2)$, and $P(\xi = 3)$.
 - b) For which n is the probability that there are at least 2 white balls in the sample greater than 0.25?
 - c) What are the limits of the probabilities computed in a) as n tends to $+\infty$?
 - d) How do the probabilities in a) change if we sample with replacement?