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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. In the base 10 number $a = \overline{374x25y}$, the digits x and y are chosen at random. What is the probability that the number is divisible by 15?

Solution: x and y can each be chosen in 10 ways, independently, so there are 100 numbers of the given form. A number is divisible by 15 exactly when it is divisible by 3 and by 5. It is divisible by 5 only if $y = 0$ or $y = 5$.

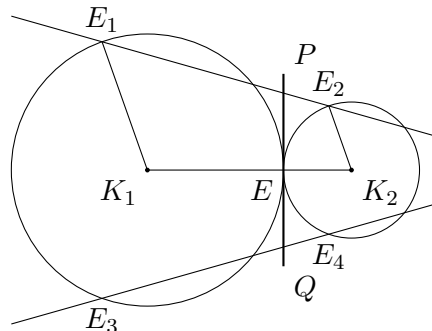
If $y = 0$, the sum of the digits is $21 + x$, so $x = 0, 3, 6$, or 9 : 4 numbers.

If $y = 5$, the sum of the digits is $26 + x$, so $x = 1, 4$, or 7 : 3 numbers.

So 7 of the 100 numbers are divisible by 15, and the probability is 0.07.

2. Two circles with different radii touch each other externally. Prove that the segment of the common internal tangent between the two common external tangents is twice the geometric mean of the two radii.

Solution: Let the circles have centers K_1, K_2 and radii R, r , touching each other at E ; let one external tangent touch them at E_1 and E_2 , the other at E_3 and E_4 , and let the internal tangent meet the external tangents at P and Q .



The tangent segments from a point to a circle are equal, so $PE = PE_1 = PE_2$ and $QE = QE_3 = QE_4$. So $PE = \frac{E_1E_2}{2}$ and $QE = \frac{E_3E_4}{2}$, and since the figure is symmetric about K_1K_2 , $PQ = E_1E_2 = a$. In the right trapezoid $K_1E_1E_2K_2$, draw the perpendicular from K_2 to K_1E_1 , with foot T ; in the right triangle K_1TK_2 :

$$(R - r)^2 + a^2 = (R + r)^2, \quad a^2 = 4Rr, \quad PQ = a = 2\sqrt{Rr}$$

3. Graph, in the Cartesian coordinate system, the sets of points that satisfy the following equations:

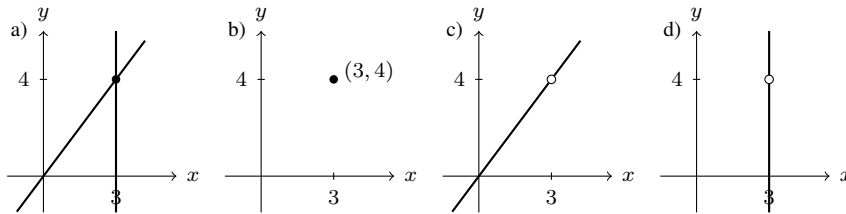
a) $(x - 3) \cdot (4x - 3y) = 0$; b) $17x^2 - 24xy + 9y^2 - 6x + 9 = 0$; c) $\frac{4x - 3y}{x - 3} = 0$; d) $\frac{x - 3}{4x - 3y} = 0$.

Solution: a) A product is 0 exactly when one of its factors is 0: $x = 3$ (and y arbitrary), or $y = \frac{4}{3}x$. The set is the union of two lines.

b) $17x^2 - 24xy + 9y^2 - 6x + 9 = (4x - 3y)^2 + (x - 3)^2 = 0$ holds only if $x - 3 = 0$ and $4x - 3y = 0$: the set is the single point $(3, 4)$.

c) $4x - 3y = 0$ and $x \neq 3$: the line $y = \frac{4}{3}x$ without the point $(3, 4)$.

d) $x = 3$ and $4x - 3y \neq 0$: the line $x = 3$ without the point $(3, 4)$.



4. Solve the following inequality over the real numbers:

$$2^x + 2^{-x} + \frac{1}{2^x + 2^{-x}} \geq -\frac{11}{3\sqrt{2}} \log_{(3-2\sqrt{2})} (3 + 2\sqrt{2})$$

Solution: The inequality is defined for every real x . The base and the argument of the logarithm are positive, the base is not 1, and they are reciprocals of each other (their product is 1), so $\log_{(3-2\sqrt{2})} (3 + 2\sqrt{2}) = -1$, and the inequality

is $2^x + 2^{-x} + \frac{1}{2^x + 2^{-x}} \geq \frac{11}{3\sqrt{2}}$.

Let $a = 2^x + 2^{-x}$; then $a \geq 2$, and

$$a + \frac{1}{a} \geq \frac{11}{3\sqrt{2}} \iff 3\sqrt{2}a^2 - 11a + 3\sqrt{2} \geq 0 \iff a \leq \frac{\sqrt{2}}{3} \text{ or } a \geq \frac{3\sqrt{2}}{2}$$

Since $a \geq 2$, only $a \geq \frac{3\sqrt{2}}{2}$ is possible: $2^x + \frac{1}{2^x} \geq \frac{3\sqrt{2}}{2}$, that is, $2(2^x)^2 - 3\sqrt{2} \cdot 2^x + 2 \geq 0$, so $2^x \leq \frac{\sqrt{2}}{2} = 2^{-\frac{1}{2}}$ or $2^x \geq \sqrt{2} = 2^{\frac{1}{2}}$. The solution is $x \leq -\frac{1}{2}$ or $x \geq \frac{1}{2}$.

Part II

5. The harmonic mean of three positive numbers is $\frac{180}{13}$, their geometric mean is $10\sqrt[3]{3}$, and their quadratic mean is $\sqrt{\frac{725}{3}}$. What is their arithmetic mean?

Solution: Let the numbers be a, b, c . By the conditions

$$\frac{3abc}{ab + ac + bc} = \frac{180}{13}, \quad abc = 3000, \quad a^2 + b^2 + c^2 = 725$$

From the first two, $ab + ac + bc = \frac{9000 \cdot 13}{180} = 650$. So $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + ac + bc) = 725 + 1300 = 2025$, and since the numbers are positive, $a + b + c = 45$. The arithmetic mean is 15.

6. a) In the quadratic equation $4px^2 - (2p^3 - 8p + 2)x + p^2 - 4 = 0$, find the values of the real parameter p for which the equation has real roots of opposite signs, neither of which is greater than $\frac{1}{2}$.
- b) Is there a value of the parameter p for which the function $y = 4px^2 - (2p^3 - 8p + 2)x + p^2 - 4$ takes its maximum at $x = 2$, and this maximum is -8 ?

Solution: a) The equation is quadratic, so $p \neq 0$. Its discriminant is

$$D = (2p^3 - 8p + 2)^2 - 16p(p^2 - 4) = (2p^3 - 8p - 2)^2 \geq 0$$

so the roots are real, and they are $x_1 = \frac{1}{2p}$ and $x_2 = \frac{p^2 - 4}{2}$. The roots have opposite signs exactly when their product $\frac{p^2 - 4}{4p}$ is negative.

If $p > 0$ and $p^2 < 4$, that is, $0 < p < 2$: $x_2 < 0 < x_1 = \frac{1}{2p}$, and $\frac{1}{2p} \leq \frac{1}{2}$ means $p \geq 1$: $1 \leq p < 2$.

If $p < 0$ and $p^2 > 4$, that is, $p < -2$: $x_1 < 0 < x_2 = \frac{p^2 - 4}{2}$, and $\frac{p^2 - 4}{2} \leq \frac{1}{2}$ means $p^2 \leq 5$: $-\sqrt{5} \leq p < -2$.

So $-\sqrt{5} \leq p < -2$ or $1 \leq p < 2$.

b) If $x = 2$, then $y = 16p - 4p^3 + 16p - 4 + p^2 - 4 = -8$ gives $4p^3 - p^2 - 32p = 0$, whose roots are 0 and $\frac{1 \pm \sqrt{513}}{8}$.

A quadratic function has a maximum only if its leading coefficient $4p$ is negative, so only $p = \frac{1 - \sqrt{513}}{8} \approx -2.71$ is

possible. But then the vertex of the parabola is at $x = \frac{2p^3 - 8p + 2}{8p} \approx 0.74$, not at $x = 2$. So there is no such p .

7. We rotate the points $A(a_1, a_2)$, $B(b_1, b_2)$, $C(-a_1, -a_2)$, and $D(-b_1, -b_2)$ around the origin by 90° in the positive direction, and then apply to the resulting points the dilation with center at the origin and ratio 2. Let the points obtained be A_1 , B_1 , C_1 , and D_1 . Prove that the midpoints of the segments AA_1 , BB_1 , CC_1 , and DD_1 determine a parallelogram, or lie on one line.

Solution: The rotation by 90° followed by the dilation takes (u, v) to $(-2v, 2u)$, so $A_1(-2a_2, 2a_1)$, $B_1(-2b_2, 2b_1)$, $C_1(2a_2, -2a_1)$, $D_1(2b_2, -2b_1)$. The midpoints are

$$F_1\left(\frac{a_1 - 2a_2}{2}, \frac{a_2 + 2a_1}{2}\right), \quad F_2\left(\frac{b_1 - 2b_2}{2}, \frac{b_2 + 2b_1}{2}\right),$$

$$F_3\left(\frac{-a_1 + 2a_2}{2}, \frac{-a_2 - 2a_1}{2}\right), \quad F_4\left(\frac{-b_1 + 2b_2}{2}, \frac{-b_2 - 2b_1}{2}\right)$$

So F_3 is the reflection of F_1 , and F_4 the reflection of F_2 , in the origin. The segments F_1F_3 and F_2F_4 bisect each other at the origin, so the four points are the vertices of a parallelogram, or (if the two segments are on one line, which happens exactly when A, B, C, D are on one line through the origin) they lie on one line.

8. How many real solutions does the equation

$$\frac{x^2 - 4x + 4}{x^2 + 2x + 3} = k$$

have, depending on the real parameter k ?

Solution: The denominator $x^2 + 2x + 3 = (x + 1)^2 + 2$ is never 0, so the equation is equivalent to $(k - 1)x^2 + (2k + 4)x + 3k - 4 = 0$.

If $k = 1$: $6x - 1 = 0$, $x = \frac{1}{6}$, one real solution.

If $k \neq 1$, the equation is quadratic, with discriminant $D = (2k + 4)^2 - 4(k - 1)(3k - 4) = -8k^2 + 44k$. If $D = 0$, that is, $k = 0$ or $k = \frac{11}{2}$, there is one real solution (a double root); if $0 < k < \frac{11}{2}$ and $k \neq 1$, two different real solutions; if $k < 0$ or $k > \frac{11}{2}$, no real solution.

(So the range of the function $x \mapsto \frac{x^2 - 4x + 4}{x^2 + 2x + 3}$ is the closed interval $\left[0, \frac{11}{2}\right]$.)

9. A box contains n white and twice as many black balls ($n \geq 3$), which cannot be told apart by touch. We draw 3 balls without replacement.

a) Let the random variable ξ be the number of white balls in the sample. Find the probabilities $P(\xi = 0)$, $P(\xi = 1)$, $P(\xi = 2)$, and $P(\xi = 3)$.

b) For which n is the probability that there are at least 2 white balls in the sample greater than 0.25?

c) What are the limits of the probabilities computed in a) as n tends to $+\infty$?

d) How do the probabilities in a) change if we sample with replacement?

Solution: a) There are $3n$ balls, and 3 can be drawn in $\binom{3n}{3}$ ways. So

$$\begin{aligned} P(\xi = 0) &= \frac{\binom{2n}{3}}{\binom{3n}{3}} = \frac{8n^2 - 12n + 4}{27n^2 - 27n + 6}, & P(\xi = 1) &= \frac{n \binom{2n}{2}}{\binom{3n}{3}} = \frac{4n^2 - 2n}{9n^2 - 9n + 2}, \\ P(\xi = 2) &= \frac{\binom{n}{2} \cdot 2n}{\binom{3n}{3}} = \frac{2n^2 - 2n}{9n^2 - 9n + 2}, & P(\xi = 3) &= \frac{\binom{n}{3}}{\binom{3n}{3}} = \frac{n^2 - 3n + 2}{3(9n^2 - 9n + 2)} \end{aligned}$$

b) $P(\xi \geq 2) = \frac{3(2n^2 - 2n) + n^2 - 3n + 2}{3(9n^2 - 9n + 2)} = \frac{7n^2 - 9n + 2}{27n^2 - 27n + 6} > \frac{1}{4}$. The denominator is positive, so this is equivalent to $n^2 - 9n + 2 > 0$, that is, $n < \frac{9 - \sqrt{73}}{2} \approx 0.23$ or $n > \frac{9 + \sqrt{73}}{2} \approx 8.77$. So $n \geq 9$.

c) The limits are $\frac{8}{27}$, $\frac{12}{27}$, $\frac{6}{27}$, and $\frac{1}{27}$.

d) With replacement, each ball drawn is white with probability $\frac{1}{3}$, independently, so $P(\xi = k) = \binom{3}{k} \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{3-k}$: $\frac{8}{27}$, $\frac{12}{27}$, $\frac{6}{27}$, $\frac{1}{27}$. These are exactly the limits computed in c).