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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. Solve the following equation: $(x - \sqrt{x-2} - 2)(x - \sqrt{x-3} - 3) = 0$.

(11 points)

Solution: The domain is $x \geq 3$. Since $x - 2 - \sqrt{x-2} = \sqrt{x-2}(\sqrt{x-2} - 1)$ and $x - 3 - \sqrt{x-3} = \sqrt{x-3}(\sqrt{x-3} - 1)$, the equation is

$$\sqrt{x-2}(\sqrt{x-2} - 1)\sqrt{x-3}(\sqrt{x-3} - 1) = 0$$

The four factors are 0 at $x = 2, 3, 3$, and 4, respectively. Taking the domain into account, the solutions are $x_1 = 3$ and $x_2 = 4$.

2. The function $f : (-1, 5] \rightarrow \mathbb{R}$, $f(x) = |(x-2)^2 - 4| - 4$ is given.

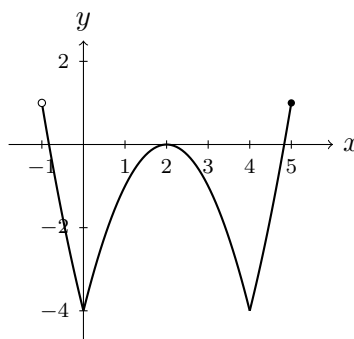
- Give the lattice points of the coordinate plane that lie on the graph of the function.
- Give the zeros of the function.
- On which intervals is the function increasing?

(12 points)

Solution: a) There are six integers in the domain (0, 1, 2, 3, 4, 5), and the function assigns an integer to each of them, so the lattice points on the graph are (0, -4), (1, -1), (2, 0), (3, -1), (4, -4), (5, 1).

b) If $(x-2)^2 \geq 4$, that is, $x \leq 0$ or $x \geq 4$, then $(x-2)^2 - 8 = 0$, so $x = 2 \pm 2\sqrt{2}$, and both values are in the domain. If $0 < x < 4$, then $-(x-2)^2 + 4 - 4 = 0$, so $x = 2$. The zeros are $2 - 2\sqrt{2}$, 2, and $2 + 2\sqrt{2}$.

c) The graph can be drawn by transforming the parabola $y = x^2$:



The function is increasing on the intervals $[0, 2]$ and $[4, 5]$.

3. Two points are given in the coordinate system: $A(1, 5)$ and $B(7, 7)$. Find the coordinates of the point P of the x -axis for which
- a) $AP = BP$; b) $AP^2 + BP^2 = 94$; c) $AP + BP$ is minimal.

(14 points)

Solution: a) The point is the intersection of the perpendicular bisector of AB with the x -axis. The perpendicular bisector passes through the midpoint $F(4, 6)$ and has normal vector $(3, 1)$, so its equation is $3x + y = 18$, and it meets the x -axis at $P_1(6, 0)$.

b) The points $P(x, y)$ with $(x - 1)^2 + (y - 5)^2 + (x - 7)^2 + (y - 7)^2 = 94$ are the points of the circle $(x - 4)^2 + (y - 6)^2 = 37$. Substituting $y = 0$ we get two solutions: $P_2(3, 0)$ and $P_3(5, 0)$.

c) Let $A'(1, -5)$ be the reflection of A in the x -axis. For every point P of the x -axis, $AP + BP = A'P + BP \geq A'B$, with equality when P is on the segment $A'B$. The line $A'B$ is $2x - y = 7$, and it meets the x -axis at $P_4\left(\frac{7}{2}, 0\right)$.

4. Solve the following system of equations:

$$2^{3x+2} = 4y^3 - 407y + 20, \quad x = \log_2(y - 4)$$

(14 points)

Solution: The domain is: x real, $y > 4$. By the second equation $2^x = y - 4$, so $2^{3x+2} = 4(2^x)^3 = 4(y - 4)^3 = 4y^3 - 48y^2 + 192y - 256$. By the first equation this is $4y^3 - 407y + 20$, so $48y^2 - 599y + 276 = 0$, whose roots are $y_1 = 12$ and $y_2 = \frac{23}{48}$. Only y_1 is in the domain, and the second equation gives $x = \log_2 8 = 3$. The solution is $x = 3$, $y = 12$.

Part II

5. The first term of a geometric sequence is a , and its ratio is q . What is the first term of the geometric sequence whose ratio is q^2 and the sum of whose first 25 terms is equal to the sum of the first 50 terms of the given sequence?

(16 points)

Solution: Let the first term of the sequence we are looking for be b . If $q = 1$, then $50a = 25b$, so $b = 2a$. If $q = -1$, then the sum of the first 50 terms is $0 = 25b$, so $b = 0$. If $q^2 \neq 1$, then by the sum formula

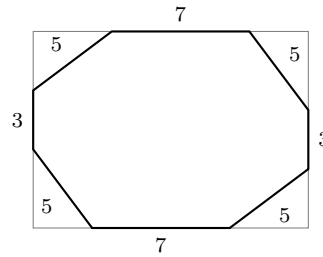
$$b \cdot \frac{(q^2)^{25} - 1}{q^2 - 1} = a \cdot \frac{q^{50} - 1}{q - 1}, \quad b = a(1 + q)$$

So in every case the first term of the new sequence is $b = a + aq$, the sum of the first two terms of the given sequence.

6. A graduating class designed its class picture board in the shape of a centrally symmetric octagon. They wanted to make the octagon from a $1 \text{ m} \times 1.4 \text{ m}$ rectangular board by cutting off four congruent right triangles at the corners of the rectangle. Then the four side lines of the rectangle are also side lines of the octagon. The sides of the octagon obtained this way, in decimeters, are in order: 7, 5, 3, 5, 7, 5, 3, 5. How much would the amount of waste change, if the octagon were cut out from the rectangle determined by the other four side lines of the octagon?

(16 points)

Solution: The hypotenuse of each of the four congruent right triangles is 5 dm. If the legs are x and $7 - x$ (the long side of the rectangle is $14 = x + 7 + (7 - x)$), then $x^2 + (7 - x)^2 = 25$, so the legs are 3 and 4 (and indeed the short side is $4 + 3 + 3 = 10$):



The other four side lines of the octagon (the sides of length 5) determine a rectangle whose corners cut off right triangles from the octagon's sides of length 7 and 3. These triangles are similar to the 3-4-5 triangle, with hypotenuses 7 and 3, so their legs are $\frac{21}{5}$, $\frac{28}{5}$ and $\frac{9}{5}$, $\frac{12}{5}$. The sides of this rectangle are

$$\frac{28}{5} + 5 + \frac{9}{5} = \frac{62}{5} \quad \text{and} \quad \frac{12}{5} + 5 + \frac{21}{5} = \frac{58}{5} \text{ dm}$$

and its area is $\frac{3596}{25} = 143.84 \text{ dm}^2$. The original rectangle has area 140 dm^2 , so the waste would be 3.84 dm^2 more.

7. Find the smallest positive value x for which $\sin x$ and $\sin 2x$ are the legs, and $\sin 3x$ the hypotenuse, of a right triangle.

(16 points)

Solution: We solve the equation $\sin^2 x + \sin^2 2x = \sin^2 3x$, that is, $\sin^2 2x = (\sin 3x - \sin x)(\sin 3x + \sin x)$. The right-hand side is

$$(2 \cos 2x \sin x)(2 \sin 2x \cos x) = 2 \sin^2 2x \cos 2x$$

so the equation is $\sin^2 2x (2 \cos 2x - 1) = 0$, and its solutions are $x = \frac{k\pi}{2}$, $x = \frac{\pi}{6} + k\pi$, $x = -\frac{\pi}{6} + k\pi$ (k an integer).

The lengths must be positive, so $\sin 2x \neq 0$, and the smallest positive suitable value is $x = \frac{\pi}{6}$. (Then the sides of the triangle are $\frac{1}{2}$, $\frac{\sqrt{3}}{2}$, and 1.)

8. A department store wants to give a present to those who buy at least seven cans of the liver pâté on sale. In the store the cans are stacked into pyramids with a square base. For example, a pyramid of four layers can be made of $16 + 9 + 4 + 1$ cans. The customers can draw an integer from 1 to 50, with equal probability, by spinning a wheel of fortune. If the number of cans in the pyramid of that many layers is divisible by 7, then the customer gets the present.

- The employees of the store built a pyramid of 16 layers. How many cans did they use?
- What is the probability that we spin a number that entitles us to the present?
- One of the customers spun a number such that the number of cans in the pyramid with that many layers was divisible by both 7 and 13. Which number could it have been?

(16 points)

Solution: a) The pyramid of n layers contains $S_n = 1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$ cans, so $S_{16} = \frac{16 \cdot 17 \cdot 33}{6} = 1496$.

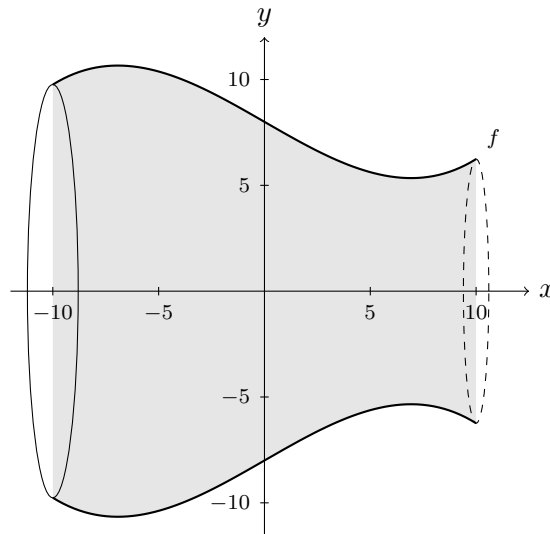
b) S_n is divisible by 7 exactly when one of the factors n , $n+1$, $2n+1$ is divisible by 7 (since 7 is a prime not dividing 6), that is, $n = 7k$, $n = 7k - 1$, or $n = 7k + 3$. In $[1, 50]$ these are 3, 6, 7, 10, 13, 14, 17, 20, 21, 24, 27, 28, 31, 34, 35, 38, 41, 42, 45, 48, 49: 21 numbers. The probability is $\frac{21}{50} = 0.42$.

c) Similarly, S_n is divisible by 13 exactly when $n = 13k$, $n = 13k - 1$, or $n = 13k + 6$; in $[1, 50]$: 6, 12, 13, 19, 25, 26, 32, 38, 39, 45. Of these, 6, 13, 38, and 45 are also on the list for 7. The customer drew one of these four numbers.

9. What is the volume of the solid of revolution obtained by rotating the graph of the cubic function $f : [-10, 10] \rightarrow \mathbb{R}$, $f(x) = 0.004x(x+12)(x-12) + 8$ around the x -axis?

(16 points)

Solution: $f(x) = 0.004(x^3 - 144x + 2000)$, and $f(x) > 0$ on $[-10, 10]$:



By the volume formula $V = \pi \int_a^b [f(x)]^2 dx$ (note that $0.004 = \frac{1}{250}$):

$$\begin{aligned}
 V &= \frac{\pi}{250^2} \int_{-10}^{10} (x^6 - 288x^4 + 4000x^3 + 20\,736x^2 - 576\,000x + 4\,000\,000) dx \\
 &= \frac{\pi}{250^2} \left[\frac{x^7}{7} - \frac{288}{5}x^5 + 1000x^4 + 6912x^3 - 288\,000x^2 + 4\,000\,000x \right]_{-10}^{10} \\
 &= \frac{\pi}{250^2} \left(2 \cdot \frac{10^7}{7} - \frac{576}{5} \cdot 10^5 + 13\,824 \cdot 10^3 + 8 \cdot 10^7 \right) \approx 4281
 \end{aligned}$$

(The odd powers cancel between -10 and 10 .)