

author: Mária Juhászné Kunstár, György Marczis, Mária Mihály, András Székely, István Tóth
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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

- Two cars start at the same time from the towns A and B , which are 24 km apart, driving toward each other. 4 minutes after they meet, the car that started from B arrives in A , and 16 minutes after they meet, the other car arrives in B . Find the speeds of the cars.

(11 points)

Solution: Let T be the meeting point, x the speed of the car from A , and y the speed of the car from B (in km/h). After the meeting, the car from A covers TB in 16 minutes, and the car from B covers TA in 4 minutes:

$$TB = \frac{16}{60}x = \frac{4}{15}x, \quad TA = \frac{4}{60}y = \frac{1}{15}y$$

Before the meeting, the car from A covered TA and the car from B covered TB , in the same time:

$$\frac{1}{15} \cdot \frac{y}{x} = \frac{4}{15} \cdot \frac{x}{y}, \quad \text{that is,} \quad \left(\frac{y}{x}\right)^2 = 4$$

So $y = 2x$ (speeds are positive). The whole distance is $\frac{1}{15} \cdot 2x + \frac{4}{15}x = 24$, so $x = 60$ and $y = 120$. The car from A drives at 60 km/h, and the car from B at 120 km/h.

- We want to distribute 1000 apples among k students ($k \geq 2$) so that the first student gets x apples, and each further student gets 3 fewer than the previous one, until all 1000 apples are given out with none left over. How many students can there be? How many apples does the last student get?
 - Next time we want to distribute 1000 apples among 16 people so that the first 11 people get the same number of apples, and after that each person always gets 50% more than the previous one. Then 15 apples are left at the end. We record the numbers of apples the people got, which gives 16 data. What are the mean and the mode of these data?

(12 points)

Solution: a) The sum of the arithmetic sequence is

$$x + (x - 3) + \dots + [x - 3(k - 1)] = \frac{2x - 3(k - 1)}{2} \cdot k = 1000$$

that is, $[2x - 3(k - 1)] \cdot k = 2000 = 2^4 \cdot 5^3$. If k is even, then $2x - 3(k - 1)$ is odd, and conversely. Taking this into account, and that the last student gets a nonnegative number of apples, the only possibilities are

$$\begin{array}{lll} 2000 = 400 \cdot 5: & k = 5, & x = 206, \text{ the last student gets } 194 \\ 2000 = 125 \cdot 16: & k = 16, & x = 85, \text{ the last student gets } 40 \\ 2000 = 80 \cdot 25: & k = 25, & x = 76, \text{ the last student gets } 4 \end{array}$$

b) The sum of the 16 data is $1000 - 15 = 985$, so the mean is $\frac{985}{16} = 61.5625$. Let the first 11 people get a apples each. The last 6 data (11th to 16th) form a geometric sequence with first term a and ratio 1.5, so

$$10a + a \cdot \frac{1.5^6 - 1}{1.5 - 1} + 15 = 1000$$

which gives $a = 32$. (The data: 32 eleven times, then 48, 72, 108, 162, 243.) The mode is 32.

3. Solve the following equations over the real numbers.

a) $\frac{x^2 + 2x + 1}{\sqrt{(x+1)^2}} + x + 1 = 0$

b) $10^{\lg 3x} = 2x - 3$

c) $\sqrt{x-1} \cdot \sqrt{x+5} = \sqrt{x^2 + 4x - 5}$

d) $\sqrt{1 - \sin^2 x} \cdot \tan x = \sin x$

(14 points)

Solution: a) The equation is undefined at $x = -1$. Since $\sqrt{(x+1)^2} = |x+1|$, the equation is $\frac{(x+1)^2}{|x+1|} = -(x+1)$, that is, $|x+1| = -(x+1)$ with $x \neq -1$. This holds exactly when $x+1 < 0$. The solutions are all real numbers $x < -1$.

b) The left side is defined only if $x > 0$, and then it equals $3x$. So $3x = 2x - 3$, $x = -3$, which is not in the domain. The equation has no solution.

c) The left side is defined only if $x \geq 1$. For these x , the equation is an identity, so the solutions are all real numbers $x \geq 1$.

d) Since $\sqrt{1 - \sin^2 x} = |\cos x|$ and $\cos x \neq 0$, the equation is $\frac{\sin x}{\cos x} |\cos x| = \sin x$.

If $\sin x = 0$: $x = k\pi$, $k \in \mathbb{Z}$, are solutions.

If $\sin x \neq 0$: $|\cos x| = \cos x$, and since $\cos x \neq 0$, $\cos x > 0$, that is, $-\frac{\pi}{2} + 2k\pi < x < \frac{\pi}{2} + 2k\pi$, $k \in \mathbb{Z}$.

So the solutions are $x = k\pi$ and $-\frac{\pi}{2} + 2k\pi < x < \frac{\pi}{2} + 2k\pi$, where k is an integer.

4. One diagonal of a parallelogram is equal to the sine of a certain angle, the other diagonal is equal to the cosine of the same angle, and the shorter side of the parallelogram is equal to the sine of twice that angle. The angle between the two diagonals is 60° . Find the exact values of the sides of the parallelogram.

(14 points)

Solution: Let the angle be α , and let the diagonals of the parallelogram $ABCD$ meet at M . The diagonals bisect each other, so the sides of the triangle MBC are $MB = \frac{\sin \alpha}{2}$ and $MC = \frac{\cos \alpha}{2}$. The shorter side $BC = \sin 2\alpha$ is opposite the 60° angle. By the Law of Cosines in the triangle MBC :

$$\left(\frac{\sin \alpha}{2}\right)^2 + \left(\frac{\cos \alpha}{2}\right)^2 - 2 \cdot \frac{\sin \alpha}{2} \cdot \frac{\cos \alpha}{2} \cdot \cos 60^\circ = \sin^2 2\alpha$$

that is, $\frac{1}{4} - \frac{1}{8} \sin 2\alpha = \sin^2 2\alpha$, so $\sin^2 2\alpha + \frac{1}{8} \sin 2\alpha - \frac{1}{4} = 0$. One root is negative, so

$$BC = \sin 2\alpha = \frac{-1 + \sqrt{65}}{16} \approx 0.44$$

In the triangle AMB the angle at M is 120° , so

$$AB^2 = \frac{1}{4} + \frac{1}{8} \sin 2\alpha = \frac{1}{4} + \frac{1}{8} \cdot \frac{-1 + \sqrt{65}}{16} = \frac{31 + \sqrt{65}}{128}$$

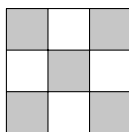
So the sides are $\frac{-1 + \sqrt{65}}{16} \approx 0.44$ and $AB = \sqrt{\frac{31 + \sqrt{65}}{128}} \approx 0.55$.

Part II

5. A biology student looks at a watery medium between two microscope slides; the medium covers a square region of side 0.9 mm. A grid of 9 squares, each with side one third of the side of this square, is projected onto the field of view; it is colored gray and white like a chessboard, with the lower left corner gray. Consider two events, when the student first looks into the microscope and sees a point-like bacterium:

A : the bacterium is in a white square;

B : the bacterium is closer to the left side of the square of medium than to its bottom side.



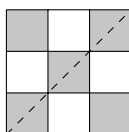
a) Find the probabilities $P(A)$, $P(B)$, and $P(\overline{A} \cdot B)$.

b) What is the probability that a spherical dust particle of radius 0.01 mm appears to have a common point with one of the gray squares? (In the microscope, the dust particle looks like a disk of radius 0.01 mm, and its position in the observed medium is random.)

(16 points)

Solution: The bacterium can be at any point of the square of medium with equal likelihood, so we use geometric probability.

a) A occurs if the bacterium is in one of the four white squares: $P(A) = \frac{4}{9}$. B occurs if the bacterium is above the dashed diagonal: $P(B) = \frac{1}{2}$.



$\overline{A} \cdot B$ occurs if the bacterium is in a gray region above the diagonal: one whole gray square and three halves. With $a = 0.9$ mm,

$$P(\overline{A} \cdot B) = \frac{\left(\frac{a}{3}\right)^2 + 3 \cdot \frac{1}{2} \left(\frac{a}{3}\right)^2}{a^2} = \frac{5}{18}$$

b) The center of the dust particle is at a random point of a square of side $b = 0.9 - 2 \cdot 0.01 = 0.88$ mm (in the extreme position the particle touches a side of the medium). The particle has no common point with any gray square exactly when its center is more than 0.01 mm from every gray square, that is, inside one of the four squares of side

$$c = \frac{0.9}{3} - 2 \cdot 0.01 = 0.28 \text{ mm}$$

in the middle of the white squares. So the probability is

$$P = \frac{b^2 - 4c^2}{b^2} = 1 - \frac{4 \cdot 0.28^2}{0.88^2} \approx 0.595$$

6. a) Consider the five-digit numbers whose last three digits are different, the sum of these three digits is 5, and none of them is a prime number. What is the sum of those numbers among these that are divisible by 8?
- b) Consider all positive integers n for which $n! + 3$ is a perfect square. How many such perfect squares are there?

(16 points)

Solution: a) The non-prime digits are 0, 1, 4, 6, 8, 9, so the last three digits can only be 0, 1, 4. Of their six orders, only 104 is divisible by 8, and a number is divisible by 8 exactly when its last three digits form a number divisible by 8. The first two digits can be any of the 90 two-digit numbers. So the numbers are 10 104, 11 104, ..., 99 104, an arithmetic sequence of 90 terms with difference 1000:

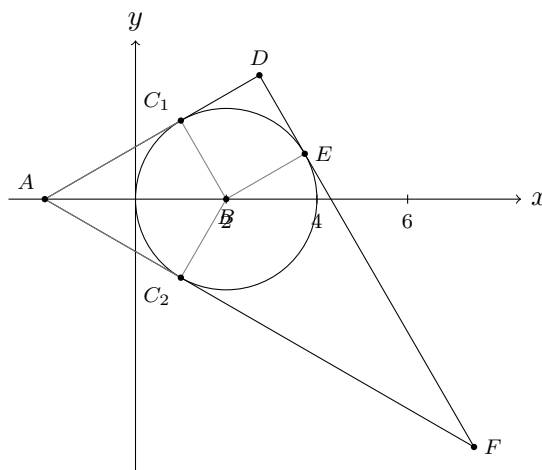
$$S_{90} = \frac{90}{2} [2 \cdot 10\,104 + 89 \cdot 1000] = 4\,914\,360$$

b) If $n \geq 5$, then the last digit of $n!$ is 0, so the last digit of $n! + 3$ is 3. No perfect square ends in 3. So we only need to check $n \leq 4$: $1! + 3 = 4$, $2! + 3 = 5$, $3! + 3 = 9$, $4! + 3 = 27$. There are two such perfect squares, 4 and 9.

7. The points $A(-2, 0)$ and $B(2, 0)$ are the endpoints of the hypotenuse of a right triangle.
- a) Find the coordinates of the third vertex if the leg opposite the vertex A is 2 units long.
- b) The circle with center B and radius 2 is the inscribed circle of a triangle that has a vertex at A , and the side opposite A is tangent to the circle at the point $E(2 + \sqrt{3}, 1)$. Find the coordinates of the other two vertices of this triangle.
- c) What are the angles of this triangle?

(16 points)

Solution:



- a) There are two such triangles, reflections of each other in the x -axis. Since $AB = 2 \cdot BC$, the angle at A is 30° , so $AC = 2\sqrt{3}$, and the third vertex is $C_1(1, \sqrt{3})$ or $C_2(1, -\sqrt{3})$.
- b) The circle is $(x - 2)^2 + y^2 = 4$; E is on it. The two sides through A are the tangent lines from A to the circle; by part a) they touch the circle at C_1 and C_2 , with direction vectors $(3, \sqrt{3})$ and $(3, -\sqrt{3})$:

$$\sqrt{3}x - 3y = -2\sqrt{3} \quad \text{and} \quad \sqrt{3}x + 3y = -2\sqrt{3}$$

The third side is tangent at E , so its normal vector is $\overrightarrow{BE} = (\sqrt{3}, 1)$, and its equation is $\sqrt{3}x + y = 2\sqrt{3} + 4$. Intersecting this line with the other two lines gives the vertices

$$D(1 + \sqrt{3}, 1 + \sqrt{3}) \quad \text{and} \quad F(4 + 2\sqrt{3}, -2 - 2\sqrt{3})$$

- c) The quadrilateral BC_1DE is a square, so the angle at D is 90° . The angle at A is $2 \cdot 30^\circ = 60^\circ$, so the angle at F is 30° .

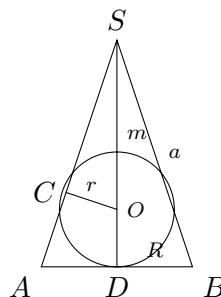
8. Let A , V , and G be the surface area and the volume of a right circular cone, and the surface area of its inscribed sphere. Prove that

$$A^2 G = 36\pi \cdot V^2$$

(16 points)

Solution: Let R be the radius of the base, m the height, a the slant height of the cone, and r the radius of the inscribed sphere. Then

$$A = \pi R(R + a), \quad V = \frac{\pi R^2 m}{3}, \quad G = 4\pi r^2$$



Cut the solids by a plane through the axis of the cone. In the isosceles triangle ABS , the triangles OCS and ADS are similar (they have two equal angles). Since $AD = R$, $DS = m$, $AS = a$, $OC = r$, and $OS = m - r$, we get $\frac{r}{R} = \frac{m - r}{a}$, so $ar = Rm - Rr$, that is, $m = \frac{r(R + a)}{R}$. Therefore

$$36\pi \cdot V^2 = 36\pi \cdot \frac{\pi^2 R^4 m^2}{9} = \frac{4\pi^3 R^4 r^2 (R + a)^2}{R^2} = \pi^2 R^2 (R + a)^2 \cdot 4\pi r^2 = A^2 G$$

9. Let $f(x) = x - \frac{2}{3}x^3$, defined on the real numbers.

- At which values of x in the interval $[-2, 1]$ does this function take its maximum and its minimum value? What are these values?
- On this interval, where is the tangent line of the graph parallel to the graph of the function $g(x) = -3x + 2007$?
- What can the rule of the function $h(x)$ be if

$$h(f(x)) = \sqrt[3]{x - \frac{2}{3}x^3}?$$

Find the composite function $f(h(x))$.

(16 points)

Solution: a) The function is differentiable everywhere, so on a closed interval it takes its extreme values at the endpoints or at the zeros of the derivative. $f'(x) = 1 - 2x^2 = 0$ at $x = \pm \frac{\sqrt{2}}{2}$.

x	$-2 \leq x < -\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2} < x < \frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2} < x \leq 1$
$f'(x)$	—	0	+	0	—
$f(x)$	$\searrow$	local min, $-\frac{\sqrt{2}}{3}$	$\nearrow$	local max, $\frac{\sqrt{2}}{3}$	$\searrow$

At the endpoints, $f(-2) = \frac{10}{3}$ and $f(1) = \frac{1}{3}$. Comparing these values: on $[-2, 1]$ the maximum is $f(-2) = \frac{10}{3}$, and the minimum is $f\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}}{3}$.

b) The tangent line has slope -3 where $1 - 2x^2 = -3$, that is, $x = \pm\sqrt{2}$. Only $x = -\sqrt{2}$ is in $[-2, 1]$, so the point of tangency is $\left(-\sqrt{2}, \frac{\sqrt{2}}{3}\right)$.

c) The expression under the cube root is exactly $f(x)$, and f takes every real value, so $h(x) = \sqrt[3]{f(x)}$. Then

$$f(h(x)) = \sqrt[3]{f(x)} - \frac{2}{3}(\sqrt[3]{f(x)})^3 = \sqrt[3]{f(x)} - \frac{2}{3}f(x)$$