

Advanced Level Practice Set - 2007 (December 2007)

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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. Solve the following equation over the real numbers.

$$\frac{2x^2 - 10x + 9}{x^2 - 4x - 12} = \frac{x^2 - x - 9}{(x - 6)(x + 2)}$$

(11 points)

2. The sides of the triangle ABC are 26, 28, and 30 units long.

a) Prove that the triangle is acute.

b) What is the longest altitude of the triangle?

c) In a plane parallel to the plane of the triangle ABC there is a triangle $A'B'C'$ that is homothetic to the triangle ABC with ratio 0.5. The distance between the two parallel planes is 12. Find the volume of the solid $ABCA'B'C'$.

(12 points)

3. a) How many four-digit numbers are divisible by 3 and end in 9?

b) If we subtract the sum of the first 50 even-indexed terms of an arithmetic sequence from the sum of its first 51 odd-indexed terms, the difference is 2007. If we subtract the sum of the first 50 odd-indexed terms of the same sequence from the sum of its first 50 even-indexed terms, the difference is 2000. Find the 101st term of the sequence.

(14 points)

4. At what angle is the segment PQ seen from the intersection point of the tangent lines drawn to the curve $y = x^2 - 2x - 2$ at its points $P(-1, 1)$ and $Q(2, -2)$? Find the coordinates of the intersection point of the tangent lines.

(14 points)

Part II

5. Let P be the point of the diagonal AC of the square $ABCD$ for which $AP = AB = 1$. The line perpendicular to AC at P meets the side BC at Q . On the line perpendicular to the plane $ABCD$ at P we measure the distance $PE = 1$.

a) Prove that the quadrilateral $ABQP$ is a kite.

b) What fraction of the area of the square is the area of the kite?

c) Find the angles between the lateral faces of the pyramid $ABCDE$ and the plane of the base.

(16 points)

6. In the expression $10^n + 14^n + 15^n + 21^n$, n denotes a positive odd integer divisible by 3.

a) Prove that for every such n the four-term sum is divisible by 1260.

b) We choose two terms of the four-term sum at random. What is the probability that the sum of the two terms is divisible by 3?

Note: The word “odd” was left out of the problem in KoMaL; the published solutions add it with an apology. Without it the statement is false, for example for $n = 6$.

(16 points)

7. Prove that the function $f(x) = 8x^3 - 6x - 1$, defined on the real numbers, has three different zeros, and the largest of them is $x_1 = \cos \frac{\pi}{9}$.

(16 points)

8. a) We cut a regular polygon with two of its diagonals into four regions: a quadrilateral, a pentagon, a hexagon, and an octagon. How many sides does the regular polygon have?

b) We colored the vertices of a regular 15-gon red, and some points in its interior green. We connected the colored points pairwise in every possible way with segments. The number of segments whose endpoints have the same color equals the number of segments whose endpoints have different colors. How many green points did we mark in the interior of the polygon?

(16 points)

9. Solve the following equation over the real numbers.

$$\sqrt{(7 + 4\sqrt{3})^x} + \sqrt{(7 - 4\sqrt{3})^x} = \frac{5}{2}$$

(16 points)