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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. Solve the following equation over the real numbers.

$$\frac{2x^2 - 10x + 9}{x^2 - 4x - 12} = \frac{x^2 - x - 9}{(x - 6)(x + 2)}$$

(11 points)

Solution: The two denominators are equal, since $x^2 - 4x - 12 = (x - 6)(x + 2)$. The domain is $x \neq -2, x \neq 6$. The numerators must be equal: $2x^2 - 10x + 9 = x^2 - x - 9$, that is,

$$x^2 - 9x + 18 = 0$$

The roots are $x_1 = 3$ and $x_2 = 6$. Only $x = 3$ is in the domain, so the solution is $x = 3$.

2. The sides of the triangle ABC are 26, 28, and 30 units long.

a) Prove that the triangle is acute.

b) What is the longest altitude of the triangle?

c) In a plane parallel to the plane of the triangle ABC there is a triangle $A'B'C'$ that is homothetic to the triangle ABC with ratio 0.5. The distance between the two parallel planes is 12. Find the volume of the solid $ABCA'B'C'$.

(12 points)

Solution: a) The largest angle γ is opposite the longest side. By the Law of Cosines,

$$30^2 = 26^2 + 28^2 - 2 \cdot 26 \cdot 28 \cdot \cos \gamma$$

so $\cos \gamma = \frac{560}{1456} \approx 0.3846 > 0$, $\gamma \approx 67.4^\circ$. The largest angle is acute, so the triangle is acute.

b) By Heron's formula, with $s = 42$: $T = \sqrt{42 \cdot 16 \cdot 14 \cdot 12} = 336$. The longest altitude belongs to the shortest side:

$$336 = \frac{26 \cdot m_a}{2}, \quad \text{so} \quad m_a = \frac{336}{13} \approx 25.85$$

Note: KoMaL gives 25.83, a rounding slip.

c) The area of the similar triangle is $t = 0.5^2 \cdot 336 = 84$. The solid is a frustum of a pyramid, so its volume is

$$V = \frac{m}{3} (T + \sqrt{Tt} + t) = \frac{12}{3} (336 + \sqrt{336 \cdot 84} + 84) = 4(336 + 168 + 84) = 2352$$

3. a) How many four-digit numbers are divisible by 3 and end in 9?

b) If we subtract the sum of the first 50 even-indexed terms of an arithmetic sequence from the sum of its first 51 odd-indexed terms, the difference is 2007. If we subtract the sum of the first 50 odd-indexed terms of the same sequence from the sum of its first 50 even-indexed terms, the difference is 2000. Find the 101st term of the sequence.

(14 points)

Solution: a) A four-digit number ending in 9 is divisible by 3 exactly when the three-digit number before the 9 is divisible by 3. There are 900 three-digit numbers, and every third one, 300 of them, is divisible by 3. So there are 300 such four-digit numbers.

b) Let the first term be a and the difference d . Then

$$a_1 + a_3 + \dots + a_{101} = 51a + 2550d$$

$$a_2 + a_4 + \dots + a_{100} = 50a + 2500d$$

$$a_1 + a_3 + \dots + a_{99} = 50a + 2450d$$

By the conditions, $a + 50d = 2007$ and $50d = 2000$. So $d = 40$, $a = 7$, and $a_{101} = a + 100d = 4007$.

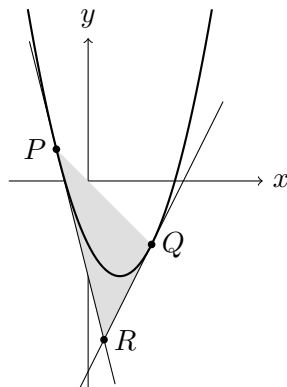
4. At what angle is the segment PQ seen from the intersection point of the tangent lines drawn to the curve $y = x^2 - 2x - 2$ at its points $P(-1, 1)$ and $Q(2, -2)$? Find the coordinates of the intersection point of the tangent lines.

(14 points)

Solution: The slope of the tangent line is given by the derivative $y' = 2x - 2$: it is -4 at P and 2 at Q . The tangent lines are

$$e_P: y - 1 = -4(x + 1), \text{ that is, } y = -4x - 3; \quad e_Q: y + 2 = 2(x - 2), \text{ that is, } y = 2x - 6$$

Their intersection point is $R\left(\frac{1}{2}, -5\right)$.



From the coordinates, $RQ = \sqrt{11.25}$, $RP = \sqrt{38.25}$, $PQ = 3\sqrt{2}$. By the Law of Cosines for the angle φ at R :

$$18 = 11.25 + 38.25 - 2 \cdot \sqrt{11.25} \cdot \sqrt{38.25} \cdot \cos \varphi$$

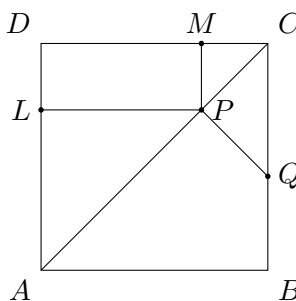
so $\cos \varphi \approx 0.7593$, and $\varphi \approx 40.6^\circ$.

Part II

5. Let P be the point of the diagonal AC of the square $ABCD$ for which $AP = AB = 1$. The line perpendicular to AC at P meets the side BC at Q . On the line perpendicular to the plane $ABCD$ at P we measure the distance $PE = 1$.
- Prove that the quadrilateral $ABQP$ is a kite.
 - What fraction of the area of the square is the area of the kite?
 - Find the angles between the lateral faces of the pyramid $ABCDE$ and the plane of the base.

(16 points)

Solution: a) $AP = AB = 1$ and $AC = \sqrt{2}$, so $PC = \sqrt{2} - 1$. The right triangle CPQ has a 45° angle at C , so $PQ = PC = \sqrt{2} - 1$, and its hypotenuse is $CQ = \sqrt{2}(\sqrt{2} - 1) = 2 - \sqrt{2}$. So $BQ = CB - CQ = 1 - (2 - \sqrt{2}) = \sqrt{2} - 1$. Since $AP = AB$ and $BQ = PQ$, the quadrilateral $ABQP$ is a kite.



- The area of the square is 1. The kite $ABQP$ has right angles at B and P , so its area is $t = AB \cdot BQ = \sqrt{2} - 1 \approx 0.41$. This is the fraction of the area of the square covered by the kite.
- By symmetry about AC , the faces ABE and ADE make the same angle α with the base, and the faces BCE and CDE make the same angle β . Let L and M be the feet of the perpendiculars from P to AD and DC . From the isosceles right triangles APL and PCM : $LP = \frac{1}{\sqrt{2}}$ and $MP = \frac{\sqrt{2} - 1}{\sqrt{2}}$. The angles are found from the right triangles LPE and MPE :

$$\tan \alpha = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}, \quad \alpha \approx 54.74^\circ; \quad \tan \beta = \frac{1}{\frac{\sqrt{2}-1}{\sqrt{2}}} = \frac{\sqrt{2}}{\sqrt{2}-1}, \quad \beta \approx 73.68^\circ$$

6. In the expression $10^n + 14^n + 15^n + 21^n$, n denotes a positive odd integer divisible by 3.
- Prove that for every such n the four-term sum is divisible by 1260.
 - We choose two terms of the four-term sum at random. What is the probability that the sum of the two terms is divisible by 3?

Note: The word "odd" was left out of the problem in KoMaL; the published solutions add it with an apology. Without it the statement is false, for example for $n = 6$.

(16 points)

Solution: a) We factor the four-term expression:

$$10^n + 14^n + 15^n + 21^n = 2^n \cdot 5^n + 2^n \cdot 7^n + 3^n \cdot 5^n + 3^n \cdot 7^n = (2^n + 3^n)(5^n + 7^n)$$

Let $n = 3k$, where k is a positive odd integer. Then

$$(2^{3k} + 3^{3k})(5^{3k} + 7^{3k}) = (8^k + 27^k)(125^k + 343^k)$$

For odd k , $u^k + v^k$ is divisible by $u + v$, so $35 \mid 8^k + 27^k$ and $468 \mid 125^k + 343^k$. Since 468 is divisible by 36 and $\gcd(35, 36) = 1$, the expression is divisible by $35 \cdot 36 = 1260$.

b) Two of the four terms can be chosen in 6 ways. Since n is odd, the sum of two terms is divisible by the sum of their bases; the sums of the bases are 24, 25, 31, 29, 35, 36. Two of these are divisible by 3, so in two cases the sum is surely divisible by 3. In the other four cases one term is divisible by 3 and the other is not, so the sum is not divisible by 3. So the probability is $\frac{2}{6} = \frac{1}{3}$.

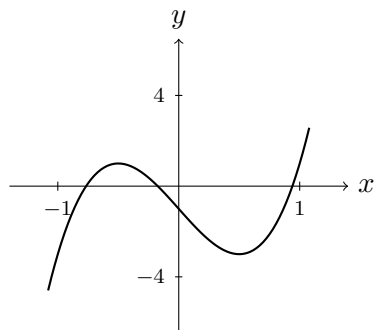
7. Prove that the function $f(x) = 8x^3 - 6x - 1$, defined on the real numbers, has three different zeros, and the largest of them is $x_1 = \cos \frac{\pi}{9}$.

(16 points)

Solution: f is continuous everywhere, and

$$f(-1) = -3, \quad f\left(-\frac{1}{2}\right) = 1, \quad f(0) = -1, \quad f(1) = 1$$

So f changes sign on each of the intervals $\left(-1, -\frac{1}{2}\right)$, $\left(-\frac{1}{2}, 0\right)$, $(0, 1)$, and it has a zero in each. A cubic polynomial has at most three zeros, so f has exactly three different zeros: two negative and one positive.



Now $\cos 3\alpha = 4\cos^3 \alpha - 3\cos \alpha$, and $3 \cdot \frac{\pi}{9} = \frac{\pi}{3}$, $\cos \frac{\pi}{3} = \frac{1}{2}$:

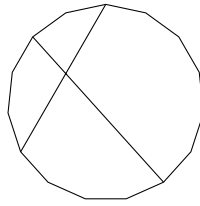
$$\frac{1}{2} = 4\cos^3 \frac{\pi}{9} - 3\cos \frac{\pi}{9}, \quad \text{that is,} \quad 8\cos^3 \frac{\pi}{9} - 6\cos \frac{\pi}{9} - 1 = 0$$

So $\cos \frac{\pi}{9}$ is a zero of f . It is positive, so it is the only positive zero, which is the largest of the three.

8. a) We cut a regular polygon with two of its diagonals into four regions: a quadrilateral, a pentagon, a hexagon, and an octagon. How many sides does the regular polygon have?
- b) We colored the vertices of a regular 15-gon red, and some points in its interior green. We connected the colored points pairwise in every possible way with segments. The number of segments whose endpoints have the same color equals the number of segments whose endpoints have different colors. How many green points did we mark in the interior of the polygon?

(16 points)

Solution: a) The two diagonals intersect each other. Each of the four regions has exactly two sides that are pieces of the diagonals, and its other sides are sides of the original polygon. Every side of the original polygon is a side of exactly one region. So the original polygon has $(4 - 2) + (5 - 2) + (6 - 2) + (8 - 2) = 2 + 3 + 4 + 6 = 15$ sides.



b) Let the number of green points be n . There are $\binom{15}{2}$ red-red segments and $\binom{n}{2}$ green-green segments, and $15n$ red-green segments. So

$$\binom{n}{2} + \binom{15}{2} = 15n, \quad \text{that is,} \quad \frac{n(n-1)}{2} + \frac{15 \cdot 14}{2} = 15n$$

which gives $n^2 - 31n + 210 = 0$, so $n = 10$ or $n = 21$. We marked 10 or 21 green points.

9. Solve the following equation over the real numbers.

$$\sqrt{(7+4\sqrt{3})^x} + \sqrt{(7-4\sqrt{3})^x} = \frac{5}{2}$$

(16 points)

Solution: Since $(2+\sqrt{3})^2 = 7+4\sqrt{3}$ and $(2-\sqrt{3})^2 = 7-4\sqrt{3}$, the equation is

$$(2+\sqrt{3})^x + (2-\sqrt{3})^x = \frac{5}{2}$$

Let $a = (2+\sqrt{3})^x$; then $(2-\sqrt{3})^x = \frac{1}{a}$, because $(2+\sqrt{3})(2-\sqrt{3}) = 1$. From $a + \frac{1}{a} = \frac{5}{2}$ we get $2a^2 - 5a + 2 = 0$, so $a = 2$ or $a = \frac{1}{2}$. So

$$x_1 = \frac{\lg 2}{\lg(2+\sqrt{3})} \approx 0.5263 \quad \text{and} \quad x_2 = -\frac{\lg 2}{\lg(2+\sqrt{3})} \approx -0.5263$$