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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

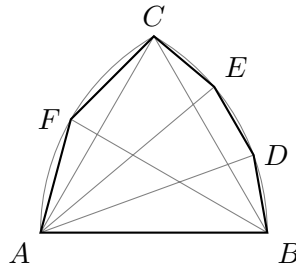
This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. In the equilateral triangle ABC , let D and E be the points dividing the minor arc BC of the circle with center A and radius AB into three equal parts, with D closer to B and E closer to C . Let F be the midpoint of the minor arc AC of the circle with center B and radius AB . Find the interior angles of the hexagon $ABDECF$.

(11 points)

Solution: Each angle of the triangle ABC is 60° , so $\angle BAD = \angle DAE = \angle EAC = 20^\circ$ and $\angle ABF = \angle FBC = 30^\circ$.



The triangles AFB and FBC are isosceles ($BA = BF = BC$) with apex angle 30° at B , so their base angles are 75° . The triangles ABD , ADE , and AEC are isosceles ($AB = AD = AE = AC$) with apex angle 20° at A , so their base angles are 80° . So the angles of the hexagon are

$$A : 75^\circ, \quad B : 80^\circ, \quad D : 2 \cdot 80^\circ = 160^\circ, \quad E : 160^\circ, \quad C : 80^\circ + 75^\circ - 60^\circ = 95^\circ, \quad F : 2 \cdot 75^\circ = 150^\circ$$

2. Solve the following equation over the real numbers.

$$\sqrt{x^2 - 2x - 15} \cdot \lg(5 - x) \cdot \sin\left(x - \frac{\pi}{3}\right) = 0$$

(12 points)

Solution: The square root is defined if $x^2 - 2x - 15 = (x + 3)(x - 5) \geq 0$, that is, $x \leq -3$ or $x \geq 5$, and the logarithm is defined if $x < 5$. So the domain of the equation is $x \leq -3$. A product is zero if one of its factors is zero.

The first factor is zero at $x = -3$ and $x = 5$, but only $x = -3$ is in the domain.

The second factor is zero at $x = 4$, which is not in the domain.

The third factor is zero at $x = \frac{\pi}{3} + k\pi$, where k is an integer. These are in the domain if $k \leq -2$.

So the solutions are $x = -3$ and $x = \frac{\pi}{3} + k\pi$, where $k \leq -2$ is an integer.

3. After a storm, four out of twelve telephone lines do not work. We try to make calls on four of the lines. What is the probability that exactly two of the four calls are successful?

(14 points)

Solution: We choose 4 of the 12 lines in $\binom{12}{4} = 495$ ways. In the favorable cases, two of the lines are among the eight working lines and two are among the four broken lines: $\binom{8}{2} \cdot \binom{4}{2} = 28 \cdot 6 = 168$ ways. So the probability is

$$p = \frac{168}{495} = \frac{56}{165} \approx 0.34$$

4. Prove that in an arithmetic sequence $\{a_n\}$

$$(x - y) a_z + (y - z) a_x = (x - z) a_y$$

where x, y , and z are natural numbers.

(14 points)

Solution: Let the first term be a_1 and the difference d , so $a_n = a_1 + (n - 1) d = (a_1 - d) + nd$. Let $c = a_1 - d$. Then

$$\begin{aligned} (x - y) a_z + (y - z) a_x - (x - z) a_y &= c[(x - y) + (y - z) - (x - z)] + d[(x - y)z + (y - z)x - (x - z)y] \\ &= c \cdot 0 + d(xz - yz + xy - xz - xy + yz) = 0 \end{aligned}$$

which proves the statement.

Part II

5. Three vertices of an isosceles trapezoid are $(-1, -3)$, $(5, -1)$, and $(2, 3)$. Find the coordinates of the fourth vertex if the product of its two coordinates is negative.

(16 points)

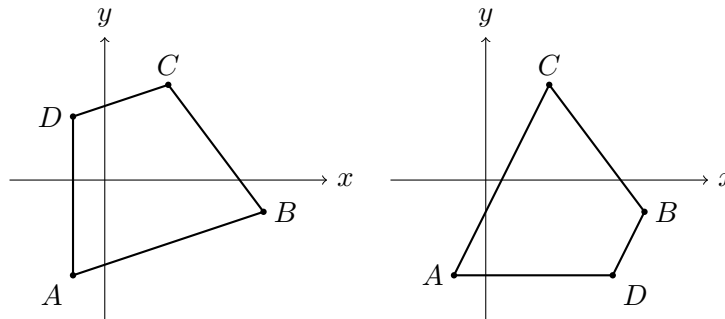
Solution: Let $A(-1, -3)$, $B(5, -1)$, $C(2, 3)$, and let D be the fourth vertex. The product of the coordinates of D is negative, so D is in Quadrant II or IV. Two of the given points form one of the parallel sides (bases), and the third point and D form the other base. An isosceles trapezoid is symmetric about the common perpendicular bisector of its bases, so D is the reflection of the third point in the perpendicular bisector of the base formed by the other two points.

If BC is a base: reflecting A gives $D(-1.6, -2.2)$, in Quadrant III. This does not satisfy the condition.

If AC is a base: the perpendicular bisector of AC is $x + 2y = 0.5$; reflecting B gives $D(4, -3)$.

If AB is a base: the perpendicular bisector of AB is $3x + y = 4$; reflecting C gives $D(-1, 2)$.

So the fourth vertex is $(-1, 2)$ or $(4, -3)$.



6. Two parallel classes of a school wrote the same test; the maximum possible score was 120 points. The average was 74 points in class A and 84 points in class B . The girls in class A averaged 71 points, and the girls in class B averaged 81 points. The average of all the girls was 79 points. The boys averaged 76 points in class A and 90 points in class B . What is the average score of all the boys in the two classes?

(16 points)

Solution: Let class A have a girls and c boys, class B have b girls and d boys, and let x be the average of all the boys. Then

$$\begin{aligned}\text{class } A: \quad 71a + 76c &= 74(a + c), \quad \text{so } c = 1.5a \\ \text{class } B: \quad 81b + 90d &= 84(b + d), \quad \text{so } d = 0.5b \\ \text{girls:} \quad 71a + 81b &= 79(a + b), \quad \text{so } b = 4a\end{aligned}$$

So $d = 0.5 \cdot 4a = 2a$, and

$$x = \frac{76c + 90d}{c + d} = \frac{76 \cdot 1.5a + 90 \cdot 2a}{1.5a + 2a} = \frac{294a}{3.5a} = 84$$

The average score of all the boys is 84 points.

7. The depth of a small stream, measured perpendicular to the direction of the flow, is given by the function $f : [0, 3] \rightarrow \mathbb{R}$, $x \mapsto \frac{x^3 - 9x}{9}$ (the units are meters).

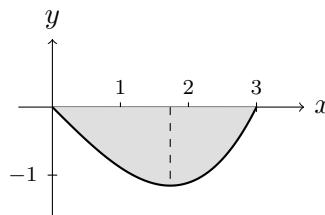
a) How far from the edge of the stream is the water the deepest?

b) What is the greatest depth of the water?

c) How much water flows through this cross section per minute if the speed of the flow is $0.5 \frac{\text{m}}{\text{s}}$?

(16 points)

Solution: a) We look for the minimum of f on $[0, 3]$. $f'(x) = \frac{1}{9}(3x^2 - 9) = \frac{1}{3}(x^2 - 3)$, which is zero on $[0, 3]$ only at $x = \sqrt{3}$, where it changes from negative to positive. So the water is deepest $\sqrt{3} \approx 1.73$ m from one edge and $3 - \sqrt{3} \approx 1.27$ m from the other edge.



b) The greatest depth is

$$\left| f(\sqrt{3}) \right| = \left| \frac{3\sqrt{3} - 9\sqrt{3}}{9} \right| = \frac{2\sqrt{3}}{3} \approx 1.15 \text{ m}$$

c) The area of the cross section is

$$\int_0^3 \frac{9x - x^3}{9} dx = \frac{1}{9} \left[\frac{9x^2}{2} - \frac{x^4}{4} \right]_0^3 = \frac{1}{9} \left(\frac{81}{2} - \frac{81}{4} \right) = 2.25 \text{ m}^2$$

In one minute the water moves $0.5 \cdot 60 = 30$ m, so $30 \cdot 2.25 = 67.5 \text{ m}^3$ of water flows through the cross section per minute.

8. The vertices of a pentagon are $A(0, 0)$, $B(12, 5)$, $C(12, 6)$, $D(10, 8)$, $E(4, 6)$.

a) How many lattice points are on the boundary of the pentagon?

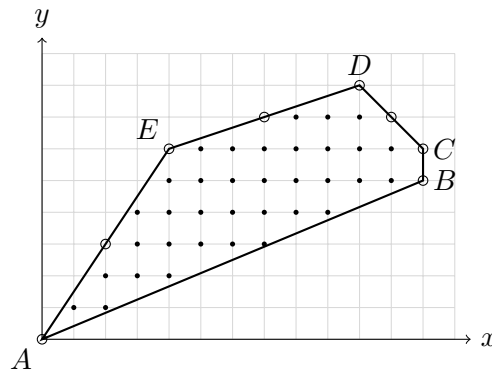
b) How many lattice points are in the interior of the pentagon?

c) The area of a polygon whose vertices are lattice points can be computed by the formula $t = \frac{h + 2b - 2}{2}$, where h is the number of lattice points on the boundary and b is the number of lattice points in the interior. What is the area of the pentagon according to this formula?

d) Find the area of the pentagon without using the formula above.

(16 points)

Solution: a) There are no lattice points inside the sides AB and BC , and there is one inside each of the sides CD , DE , and AE : $(11, 7)$, $(7, 7)$, $(2, 3)$. Together with the vertices there are 8 lattice points on the boundary.



b) Counting row by row, at the heights $y = 1, 2, 3, 4, 5, 6, 7$ there are 2, 3, 5, 7, 8, 7, 3 interior lattice points, 35 in total.

c) With $h = 8$ and $b = 35$:

$$t = \frac{8 + 2 \cdot 35 - 2}{2} = 38$$

d) The pentagon lies in the 12 by 8 rectangle with vertices $(0, 0)$ and $(12, 8)$. We cut off four right triangles and one small rectangle:

$$t = 96 - 30 - 2 - 6 - 12 - 8 = 38$$

9. Solve the following equation over the real numbers.

$$\log_3 \log_2 (4x^2) + \log_{\frac{1}{3}} \log_{\frac{1}{2}} \frac{1}{x} = 1$$

(16 points)

Solution: Since $\log_{\frac{1}{2}} \frac{1}{x} = \log_2 x$, the inner logarithms are defined and positive if $x > 1$; this is the domain. Using $\log_{\frac{1}{3}} u = -\log_3 u$:

$$\begin{aligned} \log_3 \log_2 (4x^2) - \log_3 \log_2 x &= 1 \\ \log_3 \frac{\log_2 (4x^2)}{\log_2 x} &= \log_3 3 \end{aligned}$$

Since the logarithm function is one-to-one, $\log_2 (4x^2) = 3 \log_2 x$, that is, $4x^2 = x^3$. The only solution in the domain is $x = 4$.

Note: KoMaL gives the domain as $x > 0$; in fact $\log_2 x$ must be positive as well, so $x > 1$. The solution is not affected.