

author: József Czinki

translated from: KoMaL, January 2007, pages 4-5

Pdf file: KoMaL, January 2007, pages 4-5

From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. A 20 percent (by mass) salt solution (the solvent is water) is heated, and 15 percent of the solvent evaporates. If we now add 0.1 kg of salt to the solution, its concentration (percent by mass) will be the same as if we had added 0.1 kg of solvent and 0.3 kg of salt to the original solution. What was the mass of the original solution?

(11 points)

Solution: Let the mass of the original solution be x kg. It contains $0.2x$ kg salt and $0.8x$ kg water. After 15% of the water evaporates, $0.68x$ kg water remains. After adding 0.1 kg salt, the solution contains $0.2x + 0.1$ kg salt, and its mass is $0.88x + 0.1$ kg. If we add 0.1 kg water and 0.3 kg salt to the original solution, it contains $0.2x + 0.3$ kg salt, and its mass is $x + 0.4$ kg. So

$$\frac{0.2x + 0.1}{0.88x + 0.1} = \frac{0.2x + 0.3}{x + 0.4}$$

which simplifies to $12x^2 - 52x + 5 = 0$. The solutions are $x_1 \approx 4.235$ and $x_2 \approx 0.0984$. Both satisfy the conditions, so the mass of the original solution was either about 4.235 kg or about 0.0984 kg.

2. Experience shows that 3% of the button batteries produced in a factory are defective. We select 8 button batteries at random. What is the probability that among them
 - a) there is no defective battery?
 - b) there are at least two defective batteries?

(12 points)

Solution: a) $0.97^8 \approx 0.7837$.

b) We subtract the probabilities of zero and of exactly one defective battery from 1:

$$1 - \binom{8}{0} 0.97^8 \cdot 0.03^0 - \binom{8}{1} 0.97^7 \cdot 0.03^1 \approx 0.02234$$

3. Solve the following equation over the real numbers.

$$\cos^2 x \cdot (2 \cos^2 x + 1) = \frac{105}{128 \cos^4 x + 64 \cos^2 x - 64}$$

(14 points)

Solution: The equation is undefined when $128 \cos^4 x + 64 \cos^2 x - 64 = 0$, that is, $2 \cos^4 x + \cos^2 x - 1 = 0$, $\cos^2 x = \frac{1}{2}$.

So $x \neq \frac{\pi}{4} + k \cdot \frac{\pi}{2}$, where k is an integer. The denominator can be written as $64 \cos^2 x (2 \cos^2 x + 1) - 64$, so with the new unknown $y = \cos^2 x (2 \cos^2 x + 1)$ the equation becomes

$$y = \frac{105}{64y - 64}$$

that is, $64y^2 - 64y - 105 = 0$, so $y_1 = \frac{15}{8}$ and $y_2 = -\frac{7}{8}$.

The equation $\cos^2 x (2 \cos^2 x + 1) = -\frac{7}{8}$ has no solution, because the left side is never negative.

From $\cos^2 x (2 \cos^2 x + 1) = \frac{15}{8}$ we get $16 \cos^4 x + 8 \cos^2 x - 15 = 0$, so $\cos^2 x = \frac{3}{4}$ or $\cos^2 x = -\frac{5}{4}$. The second gives no solution. From $\cos^2 x = \frac{3}{4}$:

$$x = \frac{\pi}{6} + m\pi \quad \text{or} \quad x = \frac{5\pi}{6} + n\pi$$

where m and n are integers.

4. Sanyi played the Hungarian lottery (5 numbers are marked out of 90, and 5 numbers are drawn). He selected eight numbers (out of the possible 90) and, using only these, played every possible combination of five numbers.

- How many X's did he draw on the tickets altogether?
- Two of the 8 selected numbers were drawn. How many tickets with two matches did he have?
- How many tickets with three matches would he have had if 4 of the 8 selected numbers had been drawn?

(14 points)

Solution: a) The number of ways to choose 5 of the 8 numbers is $\binom{8}{5} = 56$. On 56 tickets he marked $56 \cdot 5 = 280$ X's.

b) A ticket with two matches contains both drawn numbers and three of the six numbers that were not drawn. There are $\binom{6}{3} = 20$ such tickets.

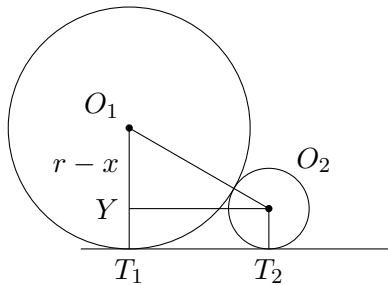
c) A ticket with three matches contains three of the four drawn numbers ($\binom{4}{3} = 4$ ways) and two of the four numbers that were not drawn ($\binom{4}{2} = 6$ ways). So he would have $4 \cdot 6 = 24$ tickets with three matches.

Part II

5. Three balls of radius r stand on a horizontal plane, each touching the other two. Find the ratio of the volume of the ball touching all three balls and the plane to the volume of the tetrahedron determined by the four centers.

(16 points)

Solution: Let O_1 be the center of one of the three equal balls and T_1 its point of contact with the plane. Let O_2 be the center of the small ball, x its radius, and T_2 its point of contact with the plane. Let Y be the foot of the perpendicular from O_2 to O_1T_1 .



The balls touch each other, so $O_1O_2 = r + x$. The centers of the three equal balls form an equilateral triangle of side $2r$, and O_2 is above its center, so $T_1T_2 = O_2Y = \frac{2}{3} \cdot \frac{\sqrt{3}}{2} \cdot 2r = \frac{2\sqrt{3}}{3}r$. By the Pythagorean theorem in the right triangle O_1O_2Y :

$$(r - x)^2 + \left(\frac{2\sqrt{3}}{3}r\right)^2 = (r + x)^2$$

so $x = \frac{r}{3}$. The base of the tetrahedron is an equilateral triangle of side $2r$, and its height is $r - x = \frac{2}{3}r$, so its volume is

$$\frac{1}{3} \cdot \frac{(2r)^2 \sqrt{3}}{4} \cdot \frac{2}{3}r = \frac{2\sqrt{3}}{9}r^3$$

The volume of the small ball is $\frac{4\pi}{3} \left(\frac{r}{3}\right)^3 = \frac{4\pi}{81}r^3$. The ratio of the volumes is

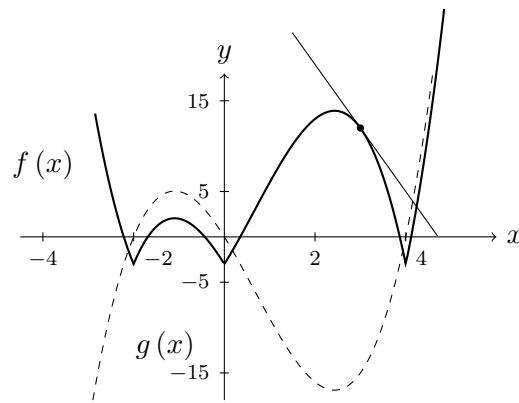
$$\frac{4\pi r^3}{81} : \frac{2\sqrt{3}r^3}{9} = \frac{2\pi\sqrt{3}}{27} \approx 0.403$$

6. Consider the function $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = |x^3 - 2x^2 - 8x| - 3$.

- Where does the function have local extrema?
- Find the equation of the tangent line of the function at the point with $x = 3$.

(16 points)

Solution: a) First consider $g(x) = x^3 - 2x^2 - 8x = x(x+2)(x-4)$. Its zeros are $x = -2, 0, 4$. Its derivative is $g'(x) = 3x^2 - 4x - 8$, and $g''(x) = 6x - 4$. The zeros of g' are $\frac{2 \pm \sqrt{28}}{3}$. By the second derivative, g has a local maximum at $\frac{2 - \sqrt{28}}{3} \approx -1.097$ and a local minimum at $\frac{2 + \sqrt{28}}{3} \approx 2.431$.



The graph of f is obtained from the graph of g by reflecting the part below the x -axis in the x -axis, then shifting down by 3 units. So f has three new extrema, and the type of one extremum of g changes. The extrema of f :

absolute minima (value -3) at $x = -2$, $x = 0$, $x = 4$;

local maxima at $x = \frac{2 - \sqrt{28}}{3}$ and $x = \frac{2 + \sqrt{28}}{3}$.

b) $g(3) = -15 < 0$, so near $x = 3$ we have $f(x) = -g(x) - 3$. Then $f(3) = 12$, and $f'(3) = -g'(3) = -7$. The tangent line is $y - 12 = -7(x - 3)$, that is, $y = -7x + 33$.

7. Someone deposits 50 000 forints (HUF) in a bank. The first bank would pay 0.4% interest per month for one year, compounded monthly. The second bank would pay 0.3% interest per month for one year, compounded monthly, and in addition, after every fourth month it increases the balance by 0.5%. The third bank pays 3% interest per half year, compounded semiannually.

a) Complete the following table (rounded to one decimal place).

	Balance at end of month 1	Balance at end of month 2	Balance at end of month 3	Balance at end of month 4	Balance at end of month 5	Balance at end of month 6
First bank						
Second bank						
Third bank						
	Balance at end of month 7	Balance at end of month 8	Balance at end of month 9	Balance at end of month 10	Balance at end of month 11	Balance at end of month 12
First bank						
Second bank						
Third bank						

b) Based on the table, for how many months is it best to put our money in the first bank, and for how many months in the second bank?

(16 points)

Solution: a)

	Balance at end of month 1	Balance at end of month 2	Balance at end of month 3	Balance at end of month 4	Balance at end of month 5	Balance at end of month 6
First bank	50 200	50 400.8	50 602.4	50 804.8	51 008.0	51 212.1
Second bank	50 150	50 300.5	50 451.4	50 855.7	51 008.3	51 161.3
Third bank	50 000	50 000	50 000	50 000	50 000	51 500
	Balance at end of month 7	Balance at end of month 8	Balance at end of month 9	Balance at end of month 10	Balance at end of month 11	Balance at end of month 12
First bank	51 416.9	51 622.6	51 829.1	52 036.4	52 244.5	52 453.5
Second bank	51 314.8	51 726.1	51 881.3	52 036.9	52 193.0	52 611.3
Third bank	51 500	51 500	51 500	51 500	51 500	53 045

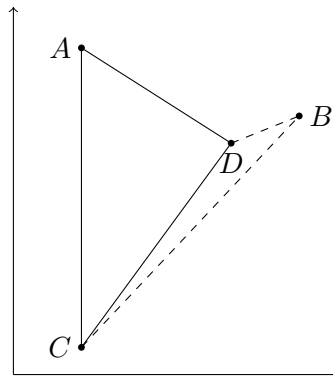
- b) The first bank is the best choice if we deposit the money for 1, 2, 3, or 11 months, and the second bank is the best if we deposit it for 4, 5, 8, 9, or 10 months. Otherwise (6, 7, or 12 months) the third bank is the best choice.
8. An ambulance airplane carrying a severely injured person was heading to the hospital of the military base at A (50, 240) (coordinates in km), but due to a technical problem over the open sea it had to make an emergency landing at the point C (50, 20). The other ambulance airplane of the service was at D (160, 170), heading to base A ; it had enough fuel for 160 km and carried no patient. What is the minimum time, counted from the emergency landing, in which the injured person can be taken to the hospital at base A by the other airplane, if fuel is available at base A and on a small island at B (210, 190), and the average speed of the airplanes is 1000 km/h? Refueling (if needed) and moving the patient to the other airplane take 10 minutes each.

(16 points)

Solution:

$$DC = \sqrt{(160 - 50)^2 + (170 - 20)^2} \approx 186 \text{ km} > 160 \text{ km}$$

so the airplane must refuel. The other distances: $AC = 220$ km, $AD \approx 130.38$ km, $DB \approx 53.85$ km, $BC \approx 233.45$ km.



On the route $D \rightarrow A \rightarrow C \rightarrow A$ the total distance is about 570.38 km, and on the route $D \rightarrow B \rightarrow C \rightarrow A$ it is about 507.30 km. Both routes include one refueling and one transfer of the patient, so the shorter route $DBCA$ is better. The flight takes $\frac{507.30}{1000} \approx 0.5073$ hours, about 30.44 minutes. Adding the 20 minutes for the transfer and refueling, the injured person can reach the hospital in about 50.44 minutes at the earliest.

9. a) Truthful (who always tells the truth), Liar (who always lies), and Uncertain (of whom we can never know whether he is lying or not) ran a race, and there was no tie. A mathematician, who watched the race but did not know them, asked them the following questions at the end of the race.

He asked the one who finished first: Who finished second? The answer was: Truthful.

He asked the one who finished second: Who are you? The answer was: Uncertain.

He asked the one who finished third: Who finished before you? The answer was: Liar.

In what order did they finish, according to the mathematician?

- b) Answer the following questions (A , B , C denote sets).

We know that $A \setminus B = C$. Is it true that $B \cup C = A$?

We know that $B \cup C = A$. Is it true that $A \setminus B = C$?

(16 points)

Solution: a) The one who finished first could not be Truthful, because he named someone else. The one who finished second could not be Truthful either, because then he would not have called himself Uncertain. So Truthful finished

third, and since he tells the truth, Liar finished second. So Uncertain won the race. The order is Uncertain, Liar, Truthful.

b) If $A \setminus B = C$, it does not follow that $B \cup C = A$. Counterexample: $A = \{1, 2, 3\}$, $B = \{1, 4, 5\}$, $C = \{2, 3\}$.

If $B \cup C = A$, it does not follow that $A \setminus B = C$. Counterexample: $A = \{1, 2, 3\}$, $B = \{1, 2\}$, $C = \{2, 3\}$.