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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. Find the exact values of the following expressions.

$$a = \lg 1000^{2007} + \lg 0.001^{2007} \quad (2 \text{ points})$$

$$b = (2005! + 2006!) \left(\frac{1}{2006!} - \frac{1}{2007!} \right) \quad (4 \text{ points})$$

$$c = \left(\frac{2}{\sqrt{2007} - 1} \right) \left(\frac{1}{1 + \sqrt{3}} + \frac{1}{\sqrt{3} + \sqrt{5}} + \dots + \frac{1}{\sqrt{2005} + \sqrt{2007}} \right) \quad (5 \text{ points})$$

Solution:

$$a = 2007 (\lg 1000 + \lg 0.001) = 2007 (3 - 3) = 0$$

$$b = 2005! (1 + 2006) \cdot \frac{2007 - 1}{2007!} = \frac{2005! \cdot 2007 \cdot 2006}{2007!} = 1$$

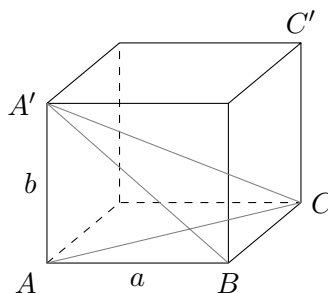
Rationalizing each denominator, $\frac{1}{\sqrt{2k-1} + \sqrt{2k+1}} = \frac{\sqrt{2k+1} - \sqrt{2k-1}}{2}$, so the sum telescopes:

$$c = \frac{2}{\sqrt{2007} - 1} \left(\frac{\sqrt{3} - 1}{2} + \frac{\sqrt{5} - \sqrt{3}}{2} + \dots + \frac{\sqrt{2007} - \sqrt{2005}}{2} \right) = \frac{2}{\sqrt{2007} - 1} \cdot \frac{\sqrt{2007} - 1}{2} = 1$$

2. The diagonal of a lateral face of a right prism with a square base is 10 cm, and the space diagonal of the prism is 12.5 cm. Find the surface area and the volume of the prism.

(12 points)

Solution: Let the base edge be a and the height b .



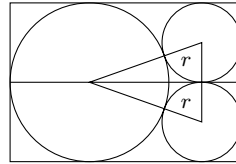
In the right triangles ABA' and ACA' (where $AC = a\sqrt{2}$):

$$a^2 + b^2 = 100, \quad 2a^2 + b^2 = 156.25$$

So $a^2 = 56.25$, $a = 7.5$ cm, and $b^2 = 43.75$, $b \approx 6.614$ cm. Then

$$V = a^2b \approx 372.06 \text{ cm}^3, \quad A = 2a^2 + 4ab \approx 310.93 \text{ cm}^2$$

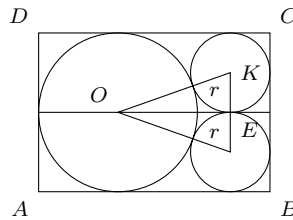
3. In a rectangular park, three flower beds in the shape of circles are made, touching each other and the sides of the rectangle as in the figure. Paths are built along the perimeter of the park, along the circles, and along all the lines through the centers of the circles drawn in the figure. The two smaller, equal flower beds have radius $r = 15$ m.



- a) What is the radius of the large circle? (2 points)
 b) What is the total length of the paths? (8 points)
 c) What percent of the area of the whole rectangle do the three circles cover? (4 points)

Solution: a) The diameter of the large circle equals the two diameters of the small circles, so $R = 2r = 30$ m.

b) Let O be the center of the large circle, K the center of one small circle, and E the point where the two small circles touch.



In the right triangle OKE , $OK = R + r = 45$ m and $KE = 15$ m, so

$$OE = \sqrt{45^2 - 15^2} = \sqrt{1800} = 30\sqrt{2} \approx 42.43 \text{ m}$$

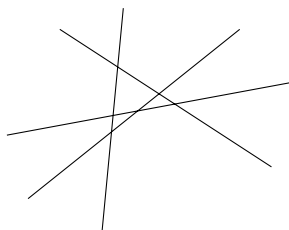
So the longer side of the rectangle is $AB = 30 + 30\sqrt{2} + 15 = 45 + 30\sqrt{2} \approx 87.43$ m, and $BC = 60$ m. The paths are the perimeter, the horizontal line through O (of length AB), the segments OK (twice), the segment joining the two small centers ($2r = 30$ m), and the three circles:

$$3AB + 2BC + 2OK + 2r + 2R\pi + 2 \cdot 2r\pi = 375 + 90\sqrt{2} + 120\pi \approx 879.3 \text{ m}$$

- c) The area of the three circles is $R^2\pi + 2r^2\pi = 1350\pi \approx 4241 \text{ m}^2$, and the area of the rectangle is $60(45 + 30\sqrt{2}) \approx 5246 \text{ m}^2$. The circles cover about 80.9% of the rectangle.

4. There are 10 lines in general position in the plane. (No two of them are parallel, and no three of them pass through the same point.)
- How many intersection points do the 10 lines have? (2 points)
 - How many non-overlapping segments and how many rays can we count on the 10 lines? (5 points)
 - We choose two of the resulting pieces (segments and rays) at random. What is the probability that the two pieces are of the same type (both segments or both rays)? (8 points)

(For example, for 4 lines in general position, as in the figure, we can count 6 intersection points, 8 segments, and 8 rays, that is, 16 pieces.)



Solution: a) Any two lines determine one intersection point, so there are $\binom{10}{2} = 45$ intersection points.

b) Each line is crossed by the 9 other lines, and the 9 intersection points cut it into 8 segments and 2 rays. So there are $10 \cdot 8 = 80$ segments and $10 \cdot 2 = 20$ rays.

c) We choose 2 of the 100 pieces in $\binom{100}{2}$ ways; two segments in $\binom{80}{2}$ ways, and two rays in $\binom{20}{2}$ ways. So the probability is

$$\frac{\binom{80}{2} + \binom{20}{2}}{\binom{100}{2}} = \frac{3160 + 190}{4950} = \frac{67}{99} \approx 0.677$$

Part II

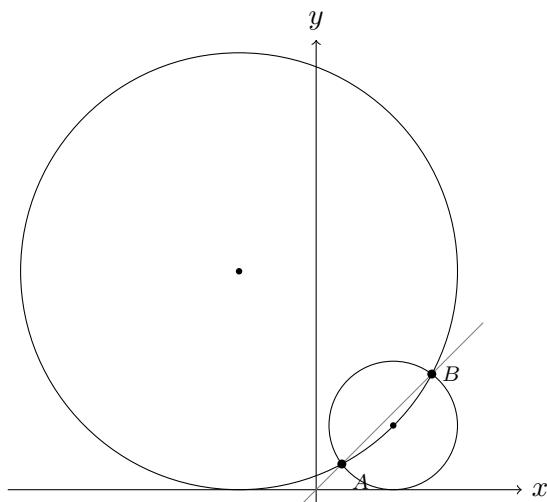
5. Given the points $A(2, 2)$ and $B(9, 9)$, find the equation of the circle that passes through A and B and is tangent to the x -axis. (16 points)

Solution: Let the center be $O(u, v)$. The circle is tangent to the x -axis, so its radius is v , and its equation is $(x - u)^2 + (y - v)^2 = v^2$. The center is on the perpendicular bisector of AB , which is $x + y = 11$, so $u = 11 - v$. The point A is on the circle:

$$(2 - (11 - v))^2 + (2 - v)^2 = v^2$$

that is, $v^2 - 22v + 85 = 0$, so $v_1 = 5$, $v_2 = 17$, and $u_1 = 6$, $u_2 = -6$. There are two such circles:

$$(x - 6)^2 + (y - 5)^2 = 25 \quad \text{and} \quad (x + 6)^2 + (y - 17)^2 = 289$$



Another way: the line AB meets the x -axis at $P(0,0)$. The tangent segment from P to the circle is the geometric mean of PA and PB : $PE^2 = PA \cdot PB = 2\sqrt{2} \cdot 9\sqrt{2} = 36$, so the points of tangency are $(6,0)$ and $(-6,0)$, which gives the same two circles.

6. We write the odd natural numbers in the following triangular table so that each row contains one more number than the previous row:

			1		
		3		5	
	7		9		11
13		15		17	19
21	23	...			

- What number is at the beginning of the 20th row? (3 points)
- In terms of n , find the number at the beginning and at the end of the n th row, and the sum of the numbers in that row. (6 points)
- What is the sum of the numbers in the row that contains the number 2007? (3 points)
- We choose one of the first 100 rows at random. What is the probability that the sum of the numbers in the chosen row is a perfect square? (4 points)

Solution: a) The rows contain 1, 2, 3, ... numbers, so the first numbers of consecutive rows differ by 2, 4, 6, ... The first number of the 20th row is

$$1 + 2 + 4 + 6 + \dots + 19 \cdot 2 = 1 + \frac{(2 + 38) \cdot 19}{2} = 381$$

- b) In the same way, the first number of the n th row is

$$1 + 2 + 4 + \dots + 2(n-1) = 1 + n(n-1) = n^2 - n + 1$$

The n numbers of the row form an arithmetic sequence with difference 2, so the last number is $n^2 - n + 1 + 2(n-1) = n^2 + n - 1$, and the sum is

$$S_n = \frac{[(n^2 - n + 1) + (n^2 + n - 1)]n}{2} = n^3$$

- We need the smallest n with $2007 \leq n^2 + n - 1$. For $n = 44$, $44^2 + 44 - 1 = 1979$, and for $n = 45$, $45^2 + 45 - 1 = 2069$. So 2007 is in the 45th row, and the sum of that row is $45^3 = 91\,125$.
- The sum n^3 is a perfect square exactly when n is a perfect square. Among 1, 2, ..., 100 there are 10 perfect squares, so the probability is $\frac{10}{100} = 0.1$.

7. a) Solve the equation $4^{x+1} - 13 \cdot 6^x + 9^{x+1} = 0$ over the real numbers. (7 points)
- b) Find all angles x in the interval $[0^\circ, 180^\circ]$ that satisfy the following equation.

$$2 \sin^2 x + 13 \sin x \cos x - 3 \cos^2 x = 6$$

(9 points)

Solution: a) The equation is $4 \cdot 4^x - 13 \cdot 6^x + 9 \cdot 9^x = 0$. Dividing by $9^x > 0$ and letting $y = \left(\frac{2}{3}\right)^x$:

$$4y^2 - 13y + 9 = 0$$

so $y = 1$ or $y = \frac{9}{4}$, that is, $x = 0$ or $x = -2$.

b) We write $6 = 6 \sin^2 x + 6 \cos^2 x$ on the right side and rearrange:

$$4 \sin^2 x - 13 \sin x \cos x + 9 \cos^2 x = 0$$

If $\cos x = 0$, then $\sin x = \pm 1$, which is not a solution, so we may divide by $\cos^2 x$: $4 \tan^2 x - 13 \tan x + 9 = 0$. This is the same quadratic as in part a), so $\tan x = 1$ or $\tan x = \frac{9}{4}$, and in $[0^\circ, 180^\circ]$: $x = 45^\circ$ or $x \approx 66.04^\circ$.

8. Inscribe a rectangle in the acute triangle ABC of area T as in Figure 1.

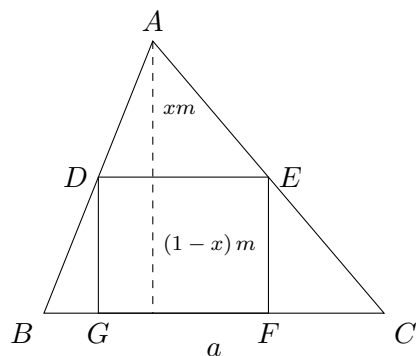


Figure 1

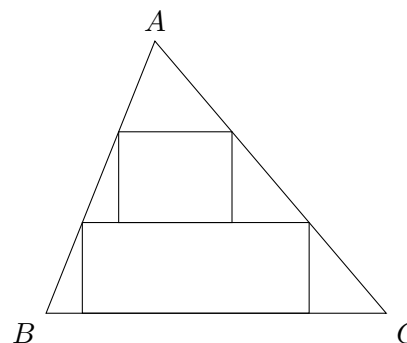


Figure 2

- a) The height of the triangle ADE from A is x times the height m of the triangle ABC from A ($0 < x < 1$). Express the area of the triangle ADE and the area of the rectangle $DEFG$ in terms of T and x . (4 points)
- b) At most what fraction of the area of the triangle ABC can the rectangle cover? (5 points)
- c) At most what fraction of the area of the triangle ABC can the two rectangles inscribed as in Figure 2 cover? (7 points)

Solution: a) The triangle ADE is similar to the triangle ABC with ratio x , so its area is $x^2 T$. With $a = BC$, we have $DE = xa$ and $DG = (1-x)m$, so the area of the rectangle is $xa(1-x)m = 2x(1-x)T$.

b) $x(1-x) = -\left(x - \frac{1}{2}\right)^2 + \frac{1}{4}$ has maximum $\frac{1}{4}$ at $x = \frac{1}{2}$. So the rectangle covers at most half of the triangle, when its height is half the height of the triangle.

c) The lower rectangle has area $2x(1-x)T$, as before. By part b), the upper rectangle covers at most half of the triangle ADE , that is, $\frac{1}{2}x^2 T$. So we maximize

$$t_1 + t_2 = \left[2x(1-x) + \frac{1}{2}x^2\right]T = \left[-\frac{3}{2}\left(x - \frac{2}{3}\right)^2 + \frac{2}{3}\right]T$$

The maximum is $\frac{2}{3}T$, at $x = \frac{2}{3}$: the two rectangles cover at most two thirds of the triangle, when each rectangle has height one third of the height of the triangle.

Note: It is wrong to make the lower rectangle as large as possible first ($\frac{1}{2}T$) and then the upper one ($\frac{1}{2} \cdot \frac{1}{4}T = \frac{1}{8}T$), because this gives only $\frac{5}{8}T < \frac{2}{3}T$.

9. Find all natural numbers n that satisfy both of the following conditions.

- The decimal form of $\frac{1}{n}$ is terminating.
- n^2 has three times as many positive divisors as n .

(16 points)

Solution: The decimal form of $\frac{1}{n}$ terminates if n has no prime factors other than 2 and 5: $n = 2^x 5^y$ ($x, y \geq 0$ integers). Then n has $(x+1)(y+1)$ divisors, and $n^2 = 2^{2x} 5^{2y}$ has $(2x+1)(2y+1)$ divisors. So

$$3(x+1)(y+1) = (2x+1)(2y+1), \quad \text{that is,} \quad xy - x - y - 2 = 0, \quad (x-1)(y-1) = 3$$

Since $x, y \geq 0$, the only possibilities are $x = 2, y = 4$ or $x = 4, y = 2$. So $n = 2^2 \cdot 5^4 = 2500$ or $n = 2^4 \cdot 5^2 = 400$, and both satisfy the conditions.

Note: KoMaL's solution has a misprint here: it writes n^2 instead of n in the last line.