

author: Levente Koncz

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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. Find all integers a, b, c with the following two properties:

- i) the sum of the three numbers is half of their product;
- ii) one of the numbers is equal to the sum of the other two.

(11 points)

Solution: We may assume that $a = b + c$. Then $2(a + b + c) = 4a = abc$, that is,

$$a(bc - 4) = 0$$

If $a = 0$: $b = -c$, where c is an arbitrary integer.

If $bc = 4$: we write 4 as a product of two integers, and $a = b + c$:

$$\begin{array}{lll} a = 5, & b = 4, & c = 1; & a = -5, & b = -4, & c = -1; \\ a = 5, & b = 1, & c = 4; & a = -5, & b = -1, & c = -4; \\ a = 4, & b = 2, & c = 2; & a = -4, & b = -2, & c = -2. \end{array}$$

(Since any of the three numbers can be the sum of the other two, the numbers can be listed in any order.)

2. The flight paths of two airplanes flying in the same plane are two perpendicular lines. At a given moment, one airplane is 2 km and the other is 2.5 km from the imaginary intersection point of their paths. The speed of the first airplane is 270 km/h, and that of the second is 180 km/h. The warning system of the airplanes warns the pilots if they get closer to each other than 1 km.

- a) Find after how many seconds the distance between the two airplanes is the smallest. (9 points)

- b) Will the warning system go off on the airplanes? (3 points)

Solution: a) Assume that both airplanes are flying toward the intersection point. The speeds are 75 m/s and 50 m/s, so after t seconds the airplanes are $2000 - 75t$ and $2500 - 50t$ meters from the intersection point, and their distance is

$$d = \sqrt{(2000 - 75t)^2 + (2500 - 50t)^2}$$

This is smallest when

$$d^2 = 8125t^2 - 550\,000t + 10\,250\,000$$

is smallest, that is, at $t = \frac{550\,000}{2 \cdot 8125} \approx 33.85$ seconds. (By then the first airplane has already passed the intersection point.)

- b) Substituting this t , the smallest distance is about $971 \text{ m} < 1 \text{ km}$, so the warning system goes off.

3. If we swap the second and third of the first three terms of a non-constant arithmetic sequence, we get the first three terms of a geometric sequence. The first term of the arithmetic sequence is equal to the opposite of the product of its second and third terms. Find the arithmetic sequence.

(14 points)

Solution: Let the first term be a and the difference $d \neq 0$. The terms of the arithmetic sequence are $a, a + d, a + 2d$, and those of the geometric sequence are $a, a + 2d, a + d$. So

$$(a + 2d)^2 = a(a + d), \quad \text{that is,} \quad 3ad = -4d^2, \quad a = -\frac{4}{3}d$$

By the second condition, $a = -(a + d)(a + 2d)$. Substituting $a = -\frac{4}{3}d$: $-\frac{4}{3}d = \frac{d}{3} \cdot \frac{2d}{3}$, so $d = -6$ and $a = 8$.

The arithmetic sequence is $a_n = 8 - 6(n - 1)$: 8, 2, -4, ... (The geometric sequence is 8, -4, 2, and indeed $8 = -2 \cdot (-4)$.)

4. In how many different ways can we form from a class of 30 students

a) a group of five? (2 points)

b) a group of at most five, but at least two students? (4 points)

c) a group of five, if the president of the class student council must be in it? (4 points)

d) a group of five, one member of which is chosen as the group leader? (4 points)

Solution: a) $\binom{30}{5} = 142\,506$.

b) $\binom{30}{5} + \binom{30}{4} + \binom{30}{3} + \binom{30}{2} = 174\,406$.

c) We choose the other four from the remaining 29 students: $\binom{29}{4} = 23\,751$.

d) Any of the five members can be the leader: $\binom{30}{5} \cdot 5 = 712\,530$.

Part II

5. A vessel in the shape of a right circular cone has a base radius of 12 cm and a height of 18 cm. We stand the vessel on its vertex and pour 1 liter of water into it.

a) How high does the water stand in it? (8 points)

b) How much paint would be needed to paint the outer wall of the vessel if 1 liter of paint covers 8 m^2 ? (8 points)

Solution: a) The water fills a cone similar to the vessel, so its height is 1.5 times its base radius ρ . With 1 liter = 1000 cm^3 :

$$1000 = \frac{\rho^2 \pi \cdot 1.5\rho}{3}, \quad \text{so} \quad 1.5\rho = 1.5 \cdot \sqrt[3]{\frac{3000}{1.5\pi}} \approx 12.9 \text{ cm}$$

The water stands about 12.9 cm high.

b) We need the lateral surface area of the cone. By the Pythagorean theorem, the slant height is $a = \sqrt{12^2 + 18^2} \approx 21.63 \text{ cm}$, and the lateral surface area is $A = r\pi a \approx 815.6 \text{ cm}^2$. Since 1000 cm^3 of paint covers 80 000 cm^2 , we need about $\frac{815.6}{80} \approx 10.2 \text{ cm}^3$, that is, about 10.2 milliliters of paint.

6. In the traditional Hungarian lottery (5 numbers out of 90), the chance of winning with one ticket (at least two matches) is about 2.33%. Hoping for the big prize, we send 100 randomly filled out tickets.
- What is the probability that we will have at least one winning ticket? (5 points)
 - What is the probability that we will have exactly three winning tickets? (5 points)
 - How many tickets would we have to play in a given week so that with probability at least 99% there is a winning ticket among them? (6 points)

Solution: a) $P(\text{at least one wins}) = 1 - P(\text{none wins}) = 1 - (1 - 0.0233)^{100} \approx 0.905$.

b) $P(\text{exactly three win}) = \binom{100}{3} \cdot 0.0233^3 \cdot (1 - 0.0233)^{97} \approx 0.208$.

c) We need $1 - 0.9767^n \geq 0.99$, that is, $0.9767^n \leq 0.01$. Taking logarithms (and dividing by the negative number $\lg 0.9767$):

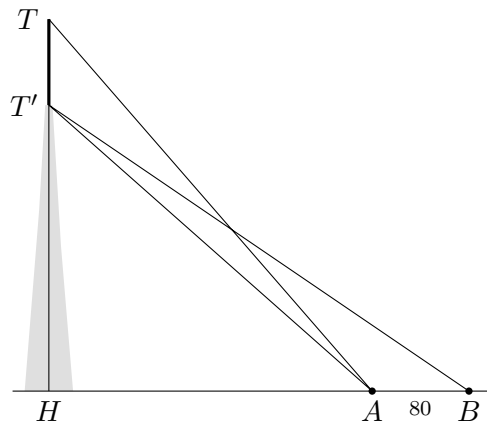
$$n \geq \frac{-2}{\lg 0.9767} \approx 195.3$$

We would have to play at least 196 tickets.

7. We want to determine the height of a radio tower standing on top of a mountain that we cannot get to. On the plain at the level of the foot of the mountain we mark points A and B , 80 meters apart, so that from B the point A and the tower are seen in the same direction (in this order). From A the foot of the tower is seen at an angle of elevation of $41^\circ 30'$ and its top at 49° ; from B the foot of the tower is seen at $34^\circ 15'$.

- How high is the mountain? (7 points)
- How high is the tower? (5 points)
- At what angle of elevation is the top of the tower seen from B ? (4 points)

Solution: Let T be the top of the tower, T' its foot, and H the projection of the tower onto the plane of the ground.



- a) $\angle T'AH = 41.5^\circ$, so $\angle T'AB = 138.5^\circ$, $\angle T'BA = 34.25^\circ$, and $\angle AT'B = 7.25^\circ$. By the Law of Sines in the triangle $T'AB$:

$$\frac{T'A}{80} = \frac{\sin 34.25^\circ}{\sin 7.25^\circ}, \quad \text{so} \quad T'A = 80 \cdot \frac{\sin 34.25^\circ}{\sin 7.25^\circ} \approx 356.8$$

From the right triangle $T'HA$: $T'H = T'A \cdot \sin 41.5^\circ \approx 236.4$. The mountain is about 236.4 m high.

- b) In the triangle $TT'A$, $\angle T'AT = 7.5^\circ$ and $\angle T'TA = 41^\circ$. By the Law of Sines:

$$\frac{TT'}{T'A} = \frac{\sin 7.5^\circ}{\sin 41^\circ}, \quad \text{so} \quad TT' = 356.8 \cdot \frac{\sin 7.5^\circ}{\sin 41^\circ} \approx 71$$

The tower is about 71 m high.

c) $HA = \frac{T'H}{\tan 41.5^\circ} \approx 267.2$, and

$$\tan \angle HBT = \frac{TH}{HB} = \frac{TT' + T'H}{HA + AB} \approx \frac{307.4}{347.2}$$

so $\angle HBT \approx 41.5^\circ$. The top of the tower is seen from B at an angle of elevation of about 41.5° .

8. In the coordinate plane, the circle k_1 is given by the equation

$$x^2 + y^2 - 4x - 2y - 20 = 0$$

a) A right triangle is inscribed in the circle k_1 ; two of its vertices are $(-2, 4)$ and $(5, 5)$. Find the coordinates of the third vertex. (6 points)

b) Find a point on the x -axis from which the tangent segments drawn to k_1 and to the circle k_2 : $(x - 12)^2 + y^2 = 4$ have equal lengths. (10 points)

Solution: a) The equation of the circle is $(x - 2)^2 + (y - 1)^2 = 25$: its center is $P(2, 1)$ and its radius is 5. Both given points are on the circle. By Thales' theorem, the hypotenuse of the right triangle is a diameter. The distance of the given points is $\sqrt{50} \neq 10$, so the segment joining them is a leg, and the third vertex is the endpoint of the diameter through one of the given points. Using the center, these are $(6, -2)$ (opposite $(-2, 4)$) and $(-1, -3)$ (opposite $(5, 5)$).

b) Let the point be $T(t, 0)$, and let S_1 and S_2 be the points of tangency on k_1 and k_2 . The center of k_2 is $U(12, 0)$, and its radius is 2. The triangles TS_1P and TS_2U have right angles at S_1 and S_2 , so by the Pythagorean theorem

$$\begin{aligned} TS_1^2 &= TP^2 - PS_1^2 = (t - 2)^2 + (0 - 1)^2 - 5^2 \\ TS_2^2 &= TU^2 - US_2^2 = (t - 12)^2 - 2^2 \end{aligned}$$

From $TS_1 = TS_2$: $(t - 2)^2 - 24 = (t - 12)^2 - 4$, so $t = 8$. The point is $(8, 0)$; the tangent segments are both $\sqrt{12}$ long.

9. a) Find the area of the region bounded by the graph of the function $f(x) = px^2 + 2x + q$ and the lines $x = 4$ and $x = 6$ (and the x -axis), if

$$\int_0^2 f(x) dx = -2 \quad \text{and} \quad \int_2^4 f(x) dx = 54$$

(9 points)

b) Find the values of k for which

$$\int_0^k f(x) dx = 0$$

(7 points)

Solution: a) The two integrals give two equations for the two unknown parameters:

$$\begin{aligned} \int_0^2 (px^2 + 2x + q) dx &= \left[\frac{px^3}{3} + x^2 + qx \right]_0^2 = \frac{8p}{3} + 4 + 2q = -2 \\ \int_2^4 (px^2 + 2x + q) dx &= \left(\frac{64p}{3} + 16 + 4q \right) - \left(\frac{8p}{3} + 4 + 2q \right) = \frac{56p}{3} + 12 + 2q = 54 \end{aligned}$$

Subtracting the first from the second: $16p + 8 = 56$, so $p = 3$, and then $q = -7$: $f(x) = 3x^2 + 2x - 7$. This is positive on $[4, 6]$, so the area is

$$\int_4^6 (3x^2 + 2x - 7) dx = [x^3 + x^2 - 7x]_4^6 = (216 + 36 - 42) - (64 + 16 - 28) = 158$$

b)

$$\int_0^k (3x^2 + 2x - 7) dx = k^3 + k^2 - 7k = k(k^2 + k - 7) = 0$$

The solutions are $k_1 = 0$, $k_2 = \frac{-1 + \sqrt{29}}{2} \approx 2.19$, and $k_3 = \frac{-1 - \sqrt{29}}{2} \approx -3.19$.