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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. Solve the following system of equations over the pairs of real numbers.

$$\begin{aligned}(x + y)(x - 4y + 11) &= x - 4y + 11 \\ (4x - y)(2x - 3y - 8) &= 16(2x - 3y - 8)\end{aligned}$$

(11 points)

2. We found at home a 120 cm long piece of a 60 cm wide canvas, from one corner of which a right isosceles triangle with legs of 30 cm had been cut off earlier. From the remaining material we want to cut out a rectangle of the largest possible area, cutting parallel to the sides of the original rectangle. What are the side lengths of the new rectangle?

(12 points)

3. The arithmetic sequence $\{a_n\} = \{9n + 8\}$ is given. From its terms we form the sequence $\{b_n\} = \{a_{n+1}^2 - a_n^2\}$.

- What is the sum of the first 2007 terms of $\{b_n\}$?
- Prove that $\{b_n\}$ is also an arithmetic sequence.
- Is 2007 a term of the sequence $\{b_n\}$?

(14 points)

4. Find the equation of the line that passes through the point $P(8, 9)$ and intersects the lines $e_1: 2x + 3y - 21 = 0$ and $e_2: 2x + 3y - 9 = 0$ at points whose y -coordinates differ by 4.

(14 points)

Part II

5. The two diagonals of a trapezoid divide it into four triangles. The triangles on the parallel sides have areas 19 cm^2 and 63 cm^2 . Find the area of the trapezoid.

(16 points)

6. We put balls with numbers written on them into an urn: the number 1 on one ball, 9 on two balls, and 3 on three balls; each ball is drawn with the same probability.

- What is the probability that drawing twice without replacement we get balls with the same number?
- We take out four balls. Can we form from them a positive integer of the form $4k + 1$ in which the product of the digits is 243 and the number is less than 100 times the sum of its digits?

(16 points)

7. Prove that in every triangle, with the usual notation for the sides and angles,

$$\frac{a \sin \beta}{\sin \left(\alpha + \frac{\gamma}{2} \right)} = \frac{2ab}{a+b} \cdot \cos \frac{\gamma}{2}$$

(16 points)

8. In a three-dimensional coordinate system, the following eight points are given: $A(4, 0, 0)$, $B(4, 4, 0)$, $C(0, 4, 0)$, $D(0, 0, 0)$, $E(4, 0, 8)$, $F(4, 4, 8)$, $G(0, 4, 8)$, $H(0, 0, 8)$.

- a) Let P be the point of the edge DH that is 1 unit from D , and let Q be the midpoint of the edge BF . What is the angle between the segments PB and PQ ?
- b) For a point R on the segment AE , the line CR intersects the line PQ at a point S . Find the coordinates of the points S and R .

(16 points)

9. The following functions are defined on the real numbers: $f(x) = 2x - 1$, $g(x) = x^2$.

- a) Find the equation of the tangent line of the function $h(x) = f(g(x)) - g(f(x))$ at the point with $x = 3$.
- b) Find the area under the curve of the function $k(x) = g(x) - f(x)$ on the interval $[1, 3]$.

(16 points)