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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. Solve the following system of equations over the pairs of real numbers.

$$\begin{aligned}(x + y)(x - 4y + 11) &= x - 4y + 11 \\ (4x - y)(2x - 3y - 8) &= 16(2x - 3y - 8)\end{aligned}$$

(11 points)

Solution: We rearrange both equations to 0 and factor:

$$\begin{aligned}(x + y - 1)(x - 4y + 11) &= 0 \\ (4x - y - 16)(2x - 3y - 8) &= 0\end{aligned}$$

We have four cases:

$$\begin{array}{llll}x + y - 1 = 0 & x + y - 1 = 0 & x - 4y + 11 = 0 & x - 4y + 11 = 0 \\4x - y - 16 = 0 & 2x - 3y - 8 = 0 & 4x - y - 16 = 0 & 2x - 3y - 8 = 0 \\x = 3.4, y = -2.4 & x = 2.2, y = -1.2 & x = 5, y = 4 & x = 13, y = 6\end{array}$$

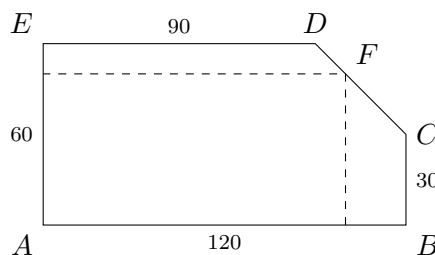
The solutions are $(3.4, -2.4)$, $(2.2, -1.2)$, $(5, 4)$, $(13, 6)$.

Note: The solutions published in KoMaL (November 2007) contain only the solution of this problem and the figures; the solutions of the other problems below are my own.

2. We found at home a 120 cm long piece of a 60 cm wide canvas, from one corner of which a right isosceles triangle with legs of 30 cm had been cut off earlier. From the remaining material we want to cut out a rectangle of the largest possible area, cutting parallel to the sides of the original rectangle. What are the side lengths of the new rectangle?

(12 points)

Solution: Let the canvas be $ABCDE$ with $AB = 120$, $AE = 60$, $BC = 30$, and $ED = 90$; the cut is the segment DC .



A rectangle of largest area has a vertex at A and the opposite vertex on the boundary. If the opposite vertex F is on the segment DC , x cm to the right of D ($0 \leq x \leq 30$), then the rectangle is $90 + x$ cm long and $60 - x$ cm wide, and its area is

$$t(x) = (90 + x)(60 - x) = 5400 - 30x - x^2$$

This is decreasing on $[0, 30]$, so the largest area is $t(0) = 5400 \text{ cm}^2$. (The full-length rectangle, 120 cm by 30 cm, is the case $x = 30$, with area only 3600 cm^2 .) So the new rectangle is 90 cm by 60 cm: we simply cut off the strip that contains the missing corner.

3. The arithmetic sequence $\{a_n\} = \{9n + 8\}$ is given. From its terms we form the sequence $\{b_n\} = \{a_{n+1}^2 - a_n^2\}$.
- What is the sum of the first 2007 terms of $\{b_n\}$?
 - Prove that $\{b_n\}$ is also an arithmetic sequence.
 - Is 2007 a term of the sequence $\{b_n\}$?

(14 points)

Solution: We start with b). Since $a_{n+1} - a_n = 9$,

$$b_n = (a_{n+1} - a_n)(a_{n+1} + a_n) = 9(18n + 25) = 162n + 225$$

so $b_{n+1} - b_n = 162$ is constant: $\{b_n\}$ is an arithmetic sequence with $b_1 = 387$ and difference 162.

a) Then $b_{2007} = 162 \cdot 2007 + 225 = 325\,359$, and

$$S_{2007} = \frac{b_1 + b_{2007}}{2} \cdot 2007 = \frac{387 + 325\,359}{2} \cdot 2007 = 326\,886\,111$$

(Or: the sum telescopes, $S_{2007} = a_{2008}^2 - a_1^2 = 18\,080^2 - 17^2$.)

c) $162n + 225 = 2007$ gives $n = 11$, so $2007 = b_{11}$ is a term of the sequence.

4. Find the equation of the line that passes through the point $P(8, 9)$ and intersects the lines $e_1: 2x + 3y - 21 = 0$ and $e_2: 2x + 3y - 9 = 0$ at points whose y -coordinates differ by 4.

(14 points)

Solution: The vertical line $x = 8$ meets e_1 at $\left(8, \frac{5}{3}\right)$ and e_2 at $\left(8, -\frac{7}{3}\right)$; the y -coordinates differ by 4, so $x = 8$ is a solution.

Otherwise the line is $y = m(x - 8) + 9$. Substituting into $2x + 3y = 21$ and $2x + 3y = 9$:

$$x_1 = 8 - \frac{12}{2 + 3m}, \quad x_2 = 8 - \frac{24}{2 + 3m}$$

(here $2 + 3m \neq 0$, since the line is not parallel to e_1). The difference of the y -coordinates is $|m(x_1 - x_2)| = \left|\frac{12m}{2 + 3m}\right| = 4$. The equation $12m = 4(2 + 3m)$ has no solution, and $12m = -4(2 + 3m)$ gives $m = -\frac{1}{3}$. This line is $y = -\frac{1}{3}(x - 8) + 9$, that is, $x + 3y = 35$; it meets e_1 at $\left(-14, \frac{49}{3}\right)$ and e_2 at $\left(-26, \frac{61}{3}\right)$.

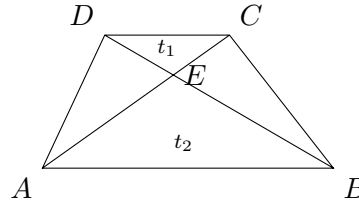
So there are two such lines: $x = 8$ and $x + 3y = 35$.

Part II

5. The two diagonals of a trapezoid divide it into four triangles. The triangles on the parallel sides have areas 19 cm^2 and 63 cm^2 . Find the area of the trapezoid.

(16 points)

Solution: Let the parallel sides be $AB = a$ and $CD = c$, the diagonals meet at E , and let $t_1 = 19$ and $t_2 = 63$ be the areas of the triangles CDE and ABE .



The triangles CDE and ABE are similar with ratio $\lambda = \frac{c}{a}$, so $\frac{t_1}{t_2} = \lambda^2$. The triangles ADE and ABE have the same height from A , and their bases are DE and BE , with $\frac{DE}{BE} = \lambda$; so the area of ADE is $\lambda t_2 = \sqrt{t_1 t_2}$. In the same way, the area of BCE is $\sqrt{t_1 t_2}$. So the area of the trapezoid is

$$T = t_1 + t_2 + 2\sqrt{t_1 t_2} = (\sqrt{t_1} + \sqrt{t_2})^2 = 19 + 63 + 2\sqrt{19 \cdot 63} = 82 + 6\sqrt{133} \approx 151.2 \text{ cm}^2$$

6. We put balls with numbers written on them into an urn: the number 1 on one ball, 9 on two balls, and 3 on three balls; each ball is drawn with the same probability.
- a) What is the probability that drawing twice without replacement we get balls with the same number?
- b) We take out four balls. Can we form from them a positive integer of the form $4k + 1$ in which the product of the digits is 243 and the number is less than 100 times the sum of its digits?

(16 points)

Solution: a) There are 6 balls, so two can be drawn in $\binom{6}{2} = 15$ ways (disregarding the order). Two balls with the same number: the two 9's (1 way) or two of the three 3's (3 ways). So the probability is $\frac{4}{15}$.

b) $243 = 3^5$, and $9 = 3^2$. The four digits must have exponents of 3 adding up to 5, so the four balls are either 9, 9, 3, 1 or 9, 3, 3, 3.

For 9, 3, 3, 3: the digit sum is 18, and every four-digit number formed from these digits is at least $3339 > 1800$. No solution.

For 9, 9, 3, 1: the digit sum is 22, so the number must be less than 2200, that is, it must start with 1: 1399, 1939, 1993. A number is of the form $4k + 1$ exactly when the number formed by its last two digits is; $99 = 4 \cdot 24 + 3$, $39 = 4 \cdot 9 + 3$, $93 = 4 \cdot 23 + 1$. So the only such number is 1993.

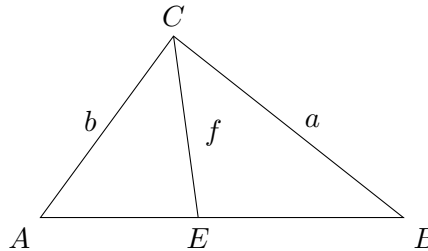
Yes, we can: the number is 1993.

7. Prove that in every triangle, with the usual notation for the sides and angles,

$$\frac{a \sin \beta}{\sin \left(\alpha + \frac{\gamma}{2} \right)} = \frac{2ab}{a+b} \cdot \cos \frac{\gamma}{2}$$

(16 points)

Solution: Let the bisector of the angle γ meet the side AB at E , and let $f = CE$.



First we compute f from areas. The bisector divides the triangle ABC into the triangles ACE and BCE , each with an angle $\frac{\gamma}{2}$ at C :

$$\frac{1}{2}ab \sin \gamma = \frac{1}{2}bf \sin \frac{\gamma}{2} + \frac{1}{2}af \sin \frac{\gamma}{2}$$

With $\sin \gamma = 2 \sin \frac{\gamma}{2} \cos \frac{\gamma}{2}$ this gives $f = \frac{2ab}{a+b} \cos \frac{\gamma}{2}$, which is the right side of the statement.

Now compute f from the triangle BCE . Its angles are $\frac{\gamma}{2}$ at C , β at B , and $180^\circ - \beta - \frac{\gamma}{2}$ at E . Since $\alpha + \beta + \gamma = 180^\circ$, we have $180^\circ - \beta - \frac{\gamma}{2} = \alpha + \frac{\gamma}{2}$. By the Law of Sines in the triangle BCE :

$$\frac{f}{\sin \beta} = \frac{a}{\sin \left(\alpha + \frac{\gamma}{2} \right)}, \quad \text{so} \quad f = \frac{a \sin \beta}{\sin \left(\alpha + \frac{\gamma}{2} \right)}$$

which is the left side of the statement. Both sides are equal to f , which proves the identity.

8. In a three-dimensional coordinate system, the following eight points are given: $A(4, 0, 0)$, $B(4, 4, 0)$, $C(0, 4, 0)$, $D(0, 0, 0)$, $E(4, 0, 8)$, $F(4, 4, 8)$, $G(0, 4, 8)$, $H(0, 0, 8)$.

a) Let P be the point of the edge DH that is 1 unit from D , and let Q be the midpoint of the edge BF . What is the angle between the segments PB and PQ ?

b) For a point R on the segment AE , the line CR intersects the line PQ at a point S . Find the coordinates of the points S and R .

(16 points)

Solution: The points are the vertices of a $4 \times 4 \times 8$ rectangular box. $P(0, 0, 1)$ and $Q(4, 4, 4)$.

a) $\overrightarrow{PB} = (4, 4, -1)$ and $\overrightarrow{PQ} = (4, 4, 3)$, so $|\overrightarrow{PB}| = \sqrt{33}$, $|\overrightarrow{PQ}| = \sqrt{41}$, and $\overrightarrow{PB} \cdot \overrightarrow{PQ} = 16 + 16 - 3 = 29$. For the angle φ between them:

$$\cos \varphi = \frac{29}{\sqrt{33} \cdot \sqrt{41}} \approx 0.7884, \quad \text{so} \quad \varphi \approx 37.96^\circ$$

b) Let $R(4, 0, r)$ with $0 \leq r \leq 8$. The points of the line PQ are $(4t, 4t, 1 + 3t)$, and the points of the line CR are $(4s, 4 - 4s, rs)$. At the intersection, $4t = 4s$ and $4t = 4 - 4s$, so $t = s = \frac{1}{2}$; then the third coordinates give $1 + \frac{3}{2} = \frac{r}{2}$, $r = 5$. So

$$S \left(2, 2, \frac{5}{2} \right) \quad \text{and} \quad R(4, 0, 5)$$

9. The following functions are defined on the real numbers: $f(x) = 2x - 1$, $g(x) = x^2$.
- a) Find the equation of the tangent line of the function $h(x) = f(g(x)) - g(f(x))$ at the point with $x = 3$.
- b) Find the area under the curve of the function $k(x) = g(x) - f(x)$ on the interval $[1, 3]$.

(16 points)

Solution: a) $h(x) = (2x^2 - 1) - (2x - 1)^2 = -2x^2 + 4x - 2 = -2(x - 1)^2$. Then $h(3) = -8$, $h'(x) = -4x + 4$, and $h'(3) = -8$. The tangent line is $y + 8 = -8(x - 3)$, that is, $y = -8x + 16$.

b) $k(x) = x^2 - 2x + 1 = (x - 1)^2 \geq 0$, so the area is

$$\int_1^3 (x - 1)^2 dx = \left[\frac{(x - 1)^3}{3} \right]_1^3 = \frac{8}{3}$$