

author: Mrs. Sándor Szlovák

translated from: KoMaL, September 2007, pages 330-331

Pdf file: KoMaL, September 2007, pages 330-331

From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. a) An urn contains 100 slips numbered from 1 to 100. We draw two slips at the same time. What is the probability that the numbers on the slips are twin primes? (Twin primes are two primes that are consecutive odd numbers.)
- b) What is the probability of drawing “triplet primes” if we draw three slips at the same time?

(11 points)

Solution: a) From 1 to 100 there are 8 pairs of twin primes: (3, 5), (5, 7), (11, 13), (17, 19), (29, 31), (41, 43), (59, 61), (71, 73). Two of the 100 slips can be chosen in $\binom{100}{2} = 4950$ ways, so

$$p = \frac{8}{4950} \approx 0.0016$$

b) Among three consecutive odd numbers, one is always divisible by 3, and the only prime divisible by 3 is 3 itself. So the only “prime triplet” is (3, 5, 7), and

$$p = \frac{1}{\binom{100}{3}} = \frac{1}{161\,700} \approx 6.184 \cdot 10^{-6}$$

2. Is the binomial coefficient $\binom{38}{19}$ divisible by 17?

(12 points)

Solution:

$$\binom{38}{19} = \frac{38!}{19! \cdot 19!} = \frac{1 \cdot 2 \cdot \dots \cdot 16 \cdot 17 \cdot 18 \cdot \dots \cdot 34 \cdot \dots \cdot 38}{(1 \cdot 2 \cdot \dots \cdot 16 \cdot 17 \cdot 18 \cdot 19)^2}$$

The numerator contains the factors 17 and $34 = 2 \cdot 17$, so it is divisible by exactly the second power of 17. The denominator is also divisible by exactly 17^2 . After simplifying by 17^2 , no factor 17 remains, and 17 is a prime. So $\binom{38}{19}$ is not divisible by 17.

3. In the equation of the circle

$$x^2 + y^2 + kx - \frac{k^2}{2}y + 1 = 0$$

find the value of the parameter k so that the circle is tangent to both coordinate axes.

(14 points)

Solution: Completing the squares:

$$\left(x + \frac{k}{2}\right)^2 + \left(y - \frac{k^2}{4}\right)^2 = \frac{k^4}{16} + \frac{k^2}{4} - 1$$

The center is $K\left(-\frac{k}{2}, \frac{k^2}{4}\right)$, and the radius is $r = \sqrt{\frac{k^4}{16} + \frac{k^2}{4} - 1}$, provided the expression under the square root is positive. The circle is tangent to both axes if the distances of the center from the axes are equal to the radius: $\left|-\frac{k}{2}\right| = \frac{k^2}{4} = r$. The equation $\left|-\frac{k}{2}\right| = \frac{k^2}{4}$ gives $k = 0$ or $k = \pm 2$. For $k = 0$ the expression under the square root is negative; for $k = \pm 2$ the radius is $r = 1$, which equals $\frac{k^2}{4}$. So $k = 2$ or $k = -2$, and the circles are

$$(x + 1)^2 + (y - 1)^2 = 1 \quad \text{and} \quad (x - 1)^2 + (y - 1)^2 = 1$$

4. a) Solve the equation $[x]^2 + 3 \cdot [x] - 10 = 0$ over the real numbers. ($[x]$ denotes the greatest integer not greater than x .)
 b) For which integers p and q is the solution set of the equation $[x]^2 + p[x] + q = 0$ a single interval of length one?

(14 points)

Solution: a) We get two values for $[x]$: $[x] = 2$ or $[x] = -5$. If $[x] = 2$, then $2 \leq x < 3$; if $[x] = -5$, then $-5 \leq x < -4$. So the solution set is $[-5, -4) \cup [2, 3)$.

b) Each integer root of the quadratic equation $t^2 + pt + q = 0$ gives one interval of length one, and other roots give nothing. If one root is an integer, then so is the other, because their sum is $-p$. So we need exactly one root, that is,

$$D = p^2 - 4q = 0$$

and then the root $-\frac{p}{2}$ must be an integer. So p is even and $q = \frac{p^2}{4}$ (a perfect square): $p = 2m$, $q = m^2$, where m is an integer. The solution set is then the interval $-m \leq x < -m + 1$.

Part II

5. Find the values of the real parameter a for which the real roots of the equation

$$x^3 + (3 - 2a)x^2 - a^2x - 3a^2 + 2a^3 = 0$$

form a geometric sequence in some order.

(16 points)

Solution: We factor the left side:

$$\begin{aligned} x^3 + (3 - 2a)x^2 - a^2x - 3a^2 + 2a^3 &= x(x^2 - a^2) + (3 - 2a)(x^2 - a^2) \\ &= (x - a)(x + a)(x + 3 - 2a) \end{aligned}$$

The roots are $x_1 = a$, $x_2 = -a$, $x_3 = 2a - 3$. There are three cases, depending on which root is the middle term of the geometric sequence.

If the middle term is $-a$: $(-a)^2 = a(2a - 3)$, so $a = 3$ or $a = 0$. For $a = 3$ the terms are $3, -3, 3$ (ratio -1). For $a = 0$ the roots are $0, 0, -3$, which do not form a geometric sequence.

If the middle term is a : $a^2 = -a(2a - 3)$, so $a = 1$ or $a = 0$. For $a = 1$ the terms are $-1, 1, -1$ (ratio -1).

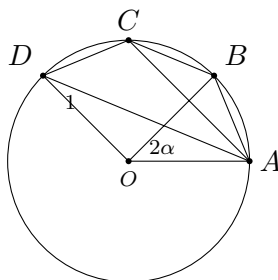
If the middle term is $2a - 3$: $(2a - 3)^2 = -a^2$, which has no real solution.

So $a = 3$ or $a = 1$.

6. In a regular polygon, A, B, C , and D are four consecutive vertices, and the length of the segment AC is the quadratic mean (root mean square) of the lengths AB and AD . How many sides does the polygon have?

(16 points)

Solution: Let the radius of the circumscribed circle be 1, and let the central angle belonging to a side be 2α . Then $AB = 2 \sin \alpha$, $AC = 2 \sin 2\alpha$, and $AD = 2 \sin 3\alpha$.



By the condition, $AC = \sqrt{\frac{AB^2 + AD^2}{2}}$, that is, $2AC^2 = AB^2 + AD^2$:

$$2 \cdot 4 \sin^2 2\alpha = 4 \sin^2 \alpha + 4 \sin^2 3\alpha, \quad \text{that is,} \quad 2 \sin^2 2\alpha = \sin^2 \alpha + \sin^2 3\alpha$$

Using $\sin 2\alpha = 2 \sin \alpha \cos \alpha$ and $\sin 3\alpha = \sin \alpha (3 - 4 \sin^2 \alpha)$:

$$8 \sin^2 \alpha \cos^2 \alpha = \sin^2 \alpha + [\sin \alpha (3 - 4 \sin^2 \alpha)]^2$$

Since $\sin^2 \alpha \neq 0$, we can divide by it: $8 \cos^2 \alpha = 1 + (3 - 4 \sin^2 \alpha)^2$, that is, $8 - 8 \sin^2 \alpha = 1 + 9 - 24 \sin^2 \alpha + 16 \sin^4 \alpha$,

$$8 \sin^4 \alpha - 8 \sin^2 \alpha + 1 = 0$$

So $\sin^2 \alpha = \frac{2 \pm \sqrt{2}}{4}$, with $\cos^2 \alpha = \frac{2 \mp \sqrt{2}}{4}$, and

$$\sin^2 2\alpha = 4 \sin^2 \alpha \cos^2 \alpha = 4 \cdot \frac{2 \pm \sqrt{2}}{4} \cdot \frac{2 \mp \sqrt{2}}{4} = \frac{1}{2}$$

We need $2\alpha = \frac{360^\circ}{n}$ with $n \geq 4$, so $2\alpha \leq 90^\circ$, and the only possibility is $2\alpha = 45^\circ$. The polygon is a regular octagon.

7. a) Solve the equation

$$2x^2 - \left[2\sqrt{2} \sin \left(\alpha + \frac{\pi}{4} \right) \right] x + \sin 2\alpha = 0$$

where α is a real parameter.

- b) Show that the function $f(\alpha) = \sin^4 \alpha + \cos^4 \alpha$, defined on the real numbers, is a transformation of the function $g(\alpha) = \cos^2 \alpha$, defined on the real numbers.

(16 points)

Solution: a) By the addition formula, $2\sqrt{2} \sin \left(\alpha + \frac{\pi}{4} \right) = 2(\sin \alpha + \cos \alpha)$, and $\sin 2\alpha = 2 \sin \alpha \cos \alpha$, so dividing by 2:

$$x^2 - (\sin \alpha + \cos \alpha)x + \sin \alpha \cos \alpha = 0$$

By Viète's formulas, the roots are $x_1 = \sin \alpha$ and $x_2 = \cos \alpha$.

- b)

$$f(\alpha) = (\sin^2 \alpha + \cos^2 \alpha)^2 - 2 \sin^2 \alpha \cos^2 \alpha = 1 - \frac{1}{2} \sin^2 2\alpha = \frac{1}{2} + \frac{1}{2} \cos^2 2\alpha = \frac{1}{2} g(2\alpha) + \frac{1}{2}$$

So the graph of f is obtained from the graph of g by a horizontal compression by a factor of 2, a vertical compression by a factor of 2, and a shift up by $\frac{1}{2}$.

8. a) Show that every member of the family of curves $y = x^3 + ax^2 + (2a - 3)x + a - 2$, where a is an arbitrary real number, passes through one point. Find the coordinates of this fixed point.

- b) How should the value of the parameter a be chosen so that the tangent line of the corresponding curve at $x_0 = 2$ passes through the point $P(-1, 0)$?

(16 points)

Solution: a) Grouping the terms with a :

$$y = x^3 - 3x - 2 + a(x^2 + 2x + 1) = (x + 1)^2(x - 2) + a(x + 1)^2$$

At $x = -1$ we get $y = 0$ for every a , so every curve passes through $(-1, 0)$. (For $x \neq -1$ the value of y depends on a , so this is the only fixed point.)

- b) Let $f(x) = x^3 + ax^2 + (2a - 3)x + a - 2$. Then $f'(x) = 3x^2 + 2ax + 2a - 3$, so the slope of the tangent line is $f'(2) = 6a + 9$, and the point of tangency is $E(2, 9a)$. The tangent line is $y - 9a = (6a + 9)(x - 2)$. It passes through $P(-1, 0)$ if $-9a = -3(6a + 9)$, that is, $a = -3$.

9. a) Find the value of the parameter
- a
- if

$$\lim_{x \rightarrow \infty} f\left(\frac{2^x - 1}{2^x + 1}\right) = 0$$

where $f(x) = ax^2 - 3x + 2$.

- b) Find the area of the region bounded by the graph of the function $f(x) = x^2 - 3x + 2$ and the x -axis.

(16 points)

Solution: a) As $x \rightarrow \infty$, $\frac{2^x - 1}{2^x + 1} = \frac{1 - 2^{-x}}{1 + 2^{-x}} \rightarrow 1$, and f is continuous, so the limit is $f(1) = a - 3 + 2 = a - 1$. So $a - 1 = 0$, $a = 1$, and $f(x) = x^2 - 3x + 2$.

- b) The zeros of f are $x_1 = 1$ and $x_2 = 2$, and f is negative between them. The area is

$$T = \left| \int_1^2 (x^2 - 3x + 2) dx \right| = \left| \left[\frac{x^3}{3} - \frac{3x^2}{2} + 2x \right]_1^2 \right| = \left| \left(\frac{8}{3} - 6 + 4 \right) - \left(\frac{1}{3} - \frac{3}{2} + 2 \right) \right| = \frac{1}{6}$$