

author: Ágnes Meszlényiné Róka

translated from: KoMaL, April 2008, pages 202-203

Pdf file: KoMaL, April 2008, pages 202-203

From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. Solve the following equations.

$$\text{a) } \frac{9}{x^2 - 16} + \frac{5}{4 - x} + \frac{13 + x}{x + 4} = -\frac{18}{4 + x} \qquad \text{b) } 2 \cos^2 x - 3 \sin x = 0$$

(11 points)

Solution: a) The domain is $x \neq \pm 4$. Rearranging:

$$\frac{9}{x^2 - 16} - \frac{5}{x - 4} + \frac{13 + x}{x + 4} + \frac{18}{x + 4} = 0$$

Multiplying both sides by $x^2 - 16$:

$$9 - 5(x + 4) + (x + 31)(x - 4) = 0, \quad \text{that is, } x^2 + 22x - 135 = 0$$

so $x_1 = 5$ and $x_2 = -27$. These are in the domain, and the steps were equivalent there, so both are solutions.

b) Since $\cos^2 x = 1 - \sin^2 x$, the equation is $2(1 - \sin^2 x) - 3 \sin x = 0$, that is, $2 \sin^2 x + 3 \sin x - 2 = 0$. The roots of this quadratic are $\frac{1}{2}$ and -2 , and $\sin x$ cannot be -2 . So $\sin x = \frac{1}{2}$:

$$x_1 = \frac{\pi}{6} + 2k_1\pi, \quad x_2 = \frac{5\pi}{6} + 2k_2\pi, \quad k_1, k_2 \in \mathbb{Z}$$

2. We draw 2 cards from a 32-card Hungarian deck (four suits: hearts, bells, leaves, and acorns, with 8 cards each). The order of drawing does not matter.

- How many different pairs of cards can we get?
- How many pairs of cards contain an acorn?
- In how many different orders can the hearts be in the full deck?
- What is the probability that in the shuffled full deck there is no heart in the first two places?

(12 points)

Solution: a) $\binom{32}{2} = \frac{32 \cdot 31}{2} = 496$ different pairs.

b) It is easier to count the pairs without an acorn: these are chosen from the other 24 cards, in $\binom{24}{2} = 276$ ways. So $496 - 276 = 220$ pairs contain at least one acorn.

- c) The 8 different hearts can be in $8! = 40\,320$ orders.
- d) There are $32!$ orders in all, and $24 \cdot 23 \cdot 30!$ favorable ones. So the probability is

$$\frac{24 \cdot 23}{32 \cdot 31} \approx 0.556$$

3. At a company, the bonus was to be divided among six people in the ratio $1 : 2 : 3 : 4 : 5 : 5$. Meanwhile it turned out that one of the employees, who would have received 25% of the money, did not meet the requirements. So they wanted to divide the 225 000 forints intended for him among the other five people so that the ratio of the amounts they receive does not change. How much money does each of the five people get?

(14 points)

Solution: Let the total amount be x forints. It was to be divided into $1 + 2 + 3 + 4 + 5 + 5 = 20$ parts, so the employees would have received $\frac{x}{20}, \frac{x}{10}, \frac{3x}{20}, \frac{x}{5}, \frac{x}{4}, \frac{x}{4}$. We know that $\frac{x}{4} = 225\,000$, so the total amount is $x = 900\,000$ forints.

Keeping the ratio $1 : 2 : 3 : 4 : 5$ for the 5 people, this amount is divided into 15 parts. So the five people get $\frac{x}{15}, \frac{2x}{15}, \frac{x}{5}, \frac{4x}{15}, \frac{x}{3}$, that is, 60 000, 120 000, 180 000, 240 000, and 300 000 forints.

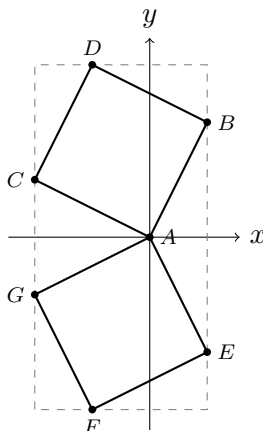
4. The vertex A of a square with side 10 units is the origin, and the slope of one of its sides is 2. The square is above the x -axis.
- a) Find the exact coordinates of the vertices of the square.
- b) We reflect the square in the x -axis. What are the coordinates of the vertices of the reflected square?
- c) What is the area of the smallest rectangle with sides parallel to the coordinate axes that contains both squares?

(14 points)

Solution: a) The side AB has length 10 and is on the line through the origin with slope 2, so the coordinates of $B(b_1, b_2)$ satisfy $b_2 = 2b_1$. By the Pythagorean theorem, $10^2 = b_1^2 + (2b_1)^2$, so $b_1 = 2\sqrt{5}$: $B(2\sqrt{5}, 4\sqrt{5})$. The vector $\overrightarrow{AC}$ is perpendicular to $\overrightarrow{AB}$ and has the same length, and the square is above the x -axis, so $\overrightarrow{AC} = (-4\sqrt{5}, 2\sqrt{5})$. Since $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{AC}$, the vertices are

$$A(0, 0), \quad B(2\sqrt{5}, 4\sqrt{5}), \quad D(-2\sqrt{5}, 6\sqrt{5}), \quad C(-4\sqrt{5}, 2\sqrt{5})$$

(in the order A, B, D, C around the square).



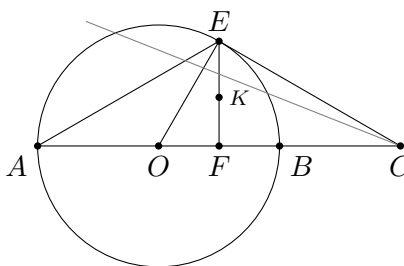
- b) Reflecting in the x -axis changes only the sign of the second coordinates: $E(2\sqrt{5}, -4\sqrt{5})$, $F(-2\sqrt{5}, -6\sqrt{5})$, $G(-4\sqrt{5}, -2\sqrt{5})$, and A stays.
- c) From the coordinates of the vertices, the side of the rectangle parallel to the x -axis is $2\sqrt{5} - (-4\sqrt{5}) = 6\sqrt{5}$, and the side parallel to the y -axis is $12\sqrt{5}$. The area of the rectangle is $T = 6\sqrt{5} \cdot 12\sqrt{5} = 360$.

Part II

5. A circle with center O and radius 5 cm is given. Let C be the point on the extension of the diameter AB beyond B , 5 cm from B . From this point we draw a tangent line to the circle; let the point of tangency be E .
- Prove that the triangle ACE is isosceles.
 - Find the side lengths and the angles of the triangle ACE .
 - What is the area of the triangle ACE ?
 - Find the exact value of the radius of the circle inscribed in the triangle ACE .

(16 points)

Solution:



- a) Since $CE \perp OE$, the triangle OCE has a right angle at E ; its shorter leg is half of the hypotenuse ($OE = 5$ cm, $OC = 10$ cm), so its acute angles are 30° (at C) and 60° (at O). The central angle $\angle EOB = 60^\circ$, so the inscribed angle on the same arc is half of it: $\angle EAC = 30^\circ$. The triangle ACE has two equal angles, so it is isosceles.
- b) Its base angles are 30° , so the angle at E is 120° . The base is $AC = 15$ cm, and the legs equal the longer leg of the right triangle OCE :

$$AE = CE = 10 \cdot \frac{\sqrt{3}}{2} = 5\sqrt{3} \approx 8.7 \text{ cm}$$

The angles of the triangle ACE are $30^\circ, 30^\circ, 120^\circ$; its base is 15 cm and its legs are about 8.7 cm.

- c) Let F be the foot of the altitude from E ; F is the midpoint of the base. The triangle $EF C$ is a 30° – 60° right triangle, so $EF = \frac{CE}{2} = \frac{5\sqrt{3}}{2}$, and

$$t = \frac{AC \cdot EF}{2} = \frac{15 \cdot \frac{5\sqrt{3}}{2}}{2} = \frac{75\sqrt{3}}{4} \approx 32.5 \text{ cm}^2$$

- d) The center K of the inscribed circle is the intersection of the angle bisectors, so the bisector from C meets the altitude EF (which is also a bisector) at K . The triangle KFC is a right triangle with a 15° angle at C , and its leg KF is the radius: $KF = FC \cdot \tan 15^\circ$. By the addition formula,

$$\tan 15^\circ = \tan(45^\circ - 30^\circ) = \frac{1 - \frac{\sqrt{3}}{3}}{1 + \frac{\sqrt{3}}{3}} = 2 - \sqrt{3}$$

So the radius is exactly $7.5(2 - \sqrt{3}) \approx 2.01$ cm.

6. a) Solve the equation $2 \cdot \left(\frac{1}{4}\right)^{3x} - \left(\frac{1}{8}\right)^{x-1} = 384$.

b) Does the equation $2 \log_{\frac{1}{4}} 3x - \log_{\frac{1}{4}} (x-1) = a$ have a real solution if the parameter a is a nonnegative real number? (16 points)

Solution: a) By the rules of exponents, $\left(\frac{1}{8}\right)^{x-1} = 8 \cdot \left(\frac{1}{8}\right)^x$ and $\left(\frac{1}{4}\right)^{3x} = \left(\frac{1}{2}\right)^{6x} = \left(\left(\frac{1}{8}\right)^x\right)^2$. With $y = \left(\frac{1}{8}\right)^x$ we get the quadratic equation

$$2y^2 - 8y - 384 = 0$$

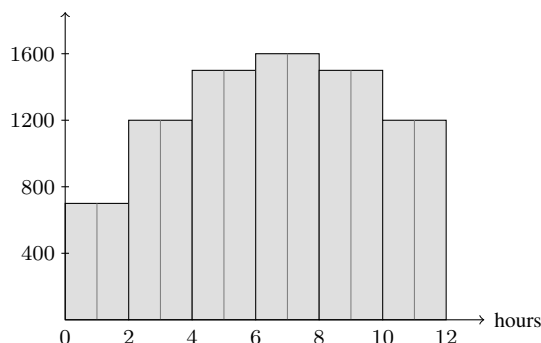
whose roots are $y_1 = 16$ and $y_2 = -12$. Only $\left(\frac{1}{8}\right)^x = 16$ is possible, that is, $2^{-3x} = 2^4$, so $x = -\frac{4}{3}$. Checking, this is indeed a solution.

b) The domain is $x > 1$. The equation can be written as

$$\log_{\frac{1}{4}} \frac{9x^2}{x-1} = a, \quad \text{that is,} \quad \frac{9x^2}{x-1} = \frac{1}{4^a}$$

so $4^a \cdot 9x^2 - x + 1 = 0$. The discriminant of this quadratic is $D = 1 - 36 \cdot 4^a$, which is negative for $a \geq 0$. So the equation has no real solution under these conditions.

7. The figure shows the result of a traffic count on a road section from midnight to noon. On the horizontal axis we show the hours, and on the vertical axis the number of passing cars, rounded to hundreds.



The traffic counter added up the vehicles passing there in two-hour periods and showed the data averaged over one hour each on a bar chart. (The area of the bars gives the number of cars passing in the given time interval.) After drawing the chart, he noticed that the six midpoints of the upper horizontal sides of the bars lie on a parabola (see the figure in the solution). According to him, this parabola shows the change of the traffic on the given road section more realistically.

a) According to the chart, how many cars passed on the given road section in the given time?

b) If we approximate the number of cars not in this way, but by the area under the parabola, what result do we get?

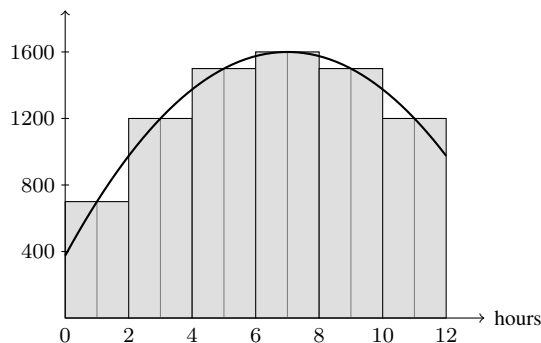
(16 points)

Solution: a) From the chart, the hourly numbers in the two-hour intervals are 700, 1200, 1500, 1600, 1500, 1200, so

$$2 \cdot (700 + 1200 + 1500 + 1600 + 1500 + 1200) = 15\,400$$

vehicles passed the traffic counter.

b) The midpoints of the tops of the bars are (1, 700), (3, 1200), (5, 1500), (7, 1600), (9, 1500), (11, 1200). The parabola opens downward with vertex (7, 1600), so it is $y = -a(x-7)^2 + 1600$; substituting (3, 1200) gives $a = 25$. So the parabola is $y = -25x^2 + 350x + 375$ (it passes through all six points).



The area under the parabola is

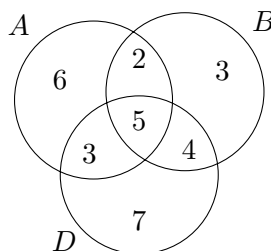
$$\int_0^{12} (-25x^2 + 350x + 375) dx = \left[-25 \cdot \frac{x^3}{3} + 350 \cdot \frac{x^2}{2} + 375x \right]_0^{12} = 15\,300$$

So by this estimate 15 300 cars passed.

8. In a mathematics competition 30 problems were given. András solved 16 problems correctly, and Balázs 14. It turned out that there were 7 problems that both of them solved correctly.
- How many problems did neither of them solve?
 - Dénes, who also took part in the competition, found that there were 5 problems that all three of them solved correctly. Dénes compared his solutions with András's and then with Balázs's, and they found that he had 8 correct solutions in common with András and 9 with Balázs. Finally they also noticed that every problem was solved correctly by at least one of them. Who among them had the most correct solutions?
 - We choose one of the 30 problems at random. What is the probability that exactly two of them solved it correctly?

(16 points)

Solution: We draw a Venn diagram; the answers can be read from it.



- $16 + 14 - 7 = 23$ problems were solved by at least one of András and Balázs, so $30 - 23 = 7$ problems were solved by neither of them.
- András and Balázs only: $7 - 5 = 2$; András and Dénes only: $8 - 5 = 3$; Balázs and Dénes only: $9 - 5 = 4$; András only: $16 - 2 - 5 - 3 = 6$; Balázs only: $14 - 2 - 5 - 4 = 3$. Every problem was solved by someone, so Dénes alone solved $30 - (6 + 2 + 3 + 5 + 3 + 4) = 7$. Dénes solved $3 + 5 + 4 + 7 = 19$ problems, the most.
- Exactly two of them solved $2 + 3 + 4 = 9$ problems correctly, so the probability is $P = \frac{9}{30} = 0.3$.

9. If both roots of the equation $x^2 + (3p - 2)x - (7p + 2) = 0$ are nonnegative, consider the two numbers as the edge lengths of two cubes. Find the value of the parameter p for which the sum of the surface areas is minimal.

(16 points)

Solution: First we check for which p the equation has real roots:

$$D = (3p - 2)^2 + 4(7p + 2) = 9p^2 + 16p + 12 = \left(3p + \frac{8}{3}\right)^2 + \frac{44}{9} > 0$$

so the equation always has two real roots. Both roots are nonnegative if $x_1 + x_2 = -(3p - 2) \geq 0$ and $x_1 x_2 = -(7p + 2) \geq 0$, that is, $p \leq -\frac{2}{7}$.

The sum of the surface areas is $A_1 + A_2 = 6x_1^2 + 6x_2^2 = 6[(x_1 + x_2)^2 - 2x_1 x_2]$, and by Viète's formulas

$$A_1 + A_2 = 6[(3p - 2)^2 + 2(7p + 2)] = 6(9p^2 + 2p + 8) = 6\left[\left(3p + \frac{1}{3}\right)^2 + \frac{71}{9}\right]$$

This quadratic function would be minimal at $p = -\frac{1}{9}$, but $p \leq -\frac{2}{7} < -\frac{1}{9}$, and on this interval the function is decreasing.

So the minimum is at $p = -\frac{2}{7}$.

Note: For $p = -\frac{2}{7}$ the roots are 0 and $\frac{20}{7}$, so one "cube" has edge 0. This is allowed by the statement (the roots are nonnegative); if we required both edges to be positive, the sum of the surface areas would have no minimum, only a greatest lower bound.