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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. Uncle Toni sold pogácsa (savory scones) and pretzels at the sports field in Pécel. He made a 25% profit on the pogácsa and a 60% profit on the pretzels. One day he sold the same amount (in value) of pogácsa as of pretzels, and so he made a 50% profit. The next day, however, he sold twice as much pogácsa as the day before, and the same amount of pretzels.

a) What percent was his profit on that day? (8 points)

b) Later Uncle Toni added gummy candy and muffins to his offer. Endre buys three items to cheer with. In how many ways can he do this if he may buy more than one of the same product? (4 points)

Solution: a) Let him receive g forints for the pogácsa and r forints for the pretzels; his profit is $0.25g$ and $0.6r$ forints. By the condition, $0.25g + 0.6r = 0.5(g + r)$, that is, $r = 2.5g$. The next day he received $2g$ for the pogácsa and r for the pretzels, and his profit rate x satisfies

$$0.25 \cdot 2g + 0.6r = x(2g + r), \quad \text{that is,} \quad 0.5g + 1.5g = x(2g + 2.5g), \quad x = \frac{2}{4.5} \approx 0.4444$$

His profit was about 44.44% on that day.

b) He chooses 3 items from 4 kinds of products with repetition allowed (combinations with repetition), so the number of ways is

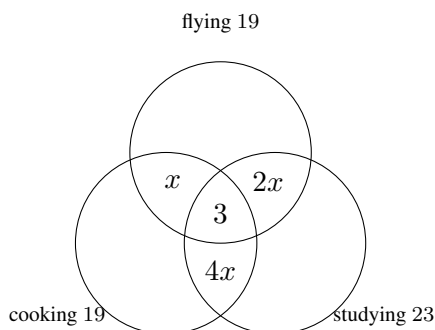
$$\binom{4+3-1}{3} = \binom{6}{3} = 20$$

2. A month on the planet U2CK3 has 41 days. According to the locals, every day is suitable for at least one of the three noble activities (flying, studying, cooking). In a month only three days are suitable for all three noble activities. The number of days suitable for flying is 19, for studying 23, and for cooking 19.

a) The numbers of days of the month that are suitable only for flying and cooking, only for flying and studying, and only for studying and cooking, are three consecutive terms of a geometric sequence with ratio 2. How many days of the month are suitable for exactly one noble activity? (6 points)

b) If we choose three of the 41 days at random, what is the probability that all three days are suitable only for cooking? (6 points)

Solution: a) Let the numbers of days suitable for exactly two activities be x (flying and cooking), $2x$ (flying and studying), and $4x$ (studying and cooking).



Adding the three sets counts each two-activity day twice and each three-activity day three times, so

$$41 = 19 + 23 + 19 - 7x - 2 \cdot 3$$

and $x = 2$. So 2, 4, and 8 days are suitable for exactly two activities, and $41 - 2 - 4 - 8 - 3 = 24$ days are suitable for exactly one noble activity.

b) The number of days suitable only for cooking is $19 - 2 - 8 - 3 = 6$. The number of favorable cases is $\binom{6}{3} = 20$, and the number of all cases is $\binom{41}{3} = 10\,660$. The probability is

$$p = \frac{20}{10\,660} \approx 0.001\,876$$

3. a) Evaluate the following integral.

$$\int \frac{1}{\cos^2 x \cdot \sin^2 x} dx$$

(8 points)

b) Find the derivative of the function $f(x) = (x^2 + 3)(x^3 + 2x - 1)$.

(5 points)

Solution: a) The integrand is defined if $\cos^2 x \cdot \sin^2 x \neq 0$, that is, $x \neq k \cdot \frac{\pi}{2}$, $k \in \mathbb{Z}$. Using $1 = \sin^2 x + \cos^2 x$:

$$\begin{aligned} \int \frac{1}{\cos^2 x \cdot \sin^2 x} dx &= \int \frac{\sin^2 x + \cos^2 x}{\cos^2 x \cdot \sin^2 x} dx = \int \frac{1}{\cos^2 x} dx + \int \frac{1}{\sin^2 x} dx \\ &= \tan x - \cot x + C \end{aligned}$$

b)

$$f'(x) = (x^5 + 5x^3 - x^2 + 6x - 3)' = 5x^4 + 15x^2 - 2x + 6$$

4. a) Graph the following function.

$$f(x) = \begin{cases} -x^2 + 4, & \text{if } x \geq 0 \\ \frac{x^2 - 5x}{x^2 - x - 20}, & \text{if } x < 0 \text{ and } x \neq -4 \end{cases}$$

(8 points)

- b) Let k be a real number. How many zeros does the function $g(x) = f(x) - k$ have?

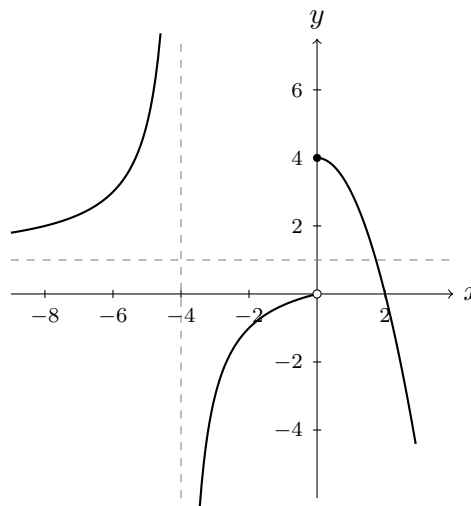
(6 points)

Note: KoMaL prints the condition as “ $x < 0$ and $x \neq 4$ ”; the excluded value is $x = -4$.

Solution: a) The domain is $x \in \mathbb{R} \setminus \{-4\}$. For $x < 0$:

$$\frac{x^2 - 5x}{x^2 - x - 20} = \frac{x(x - 5)}{(x - 5)(x + 4)} = \frac{x}{x + 4} = \frac{x + 4 - 4}{x + 4} = -4 \cdot \frac{1}{x + 4} + 1$$

So for $x < 0$ the graph is a hyperbola with asymptotes $x = -4$ and $y = 1$, and for $x \geq 0$ it is a downward parabola with vertex $(0, 4)$.



- b) The number of zeros of g is the number of solutions of $f(x) = k$. For $x < -4$ the function takes every value greater than 1 once, for $-4 < x < 0$ every negative value once, and for $x \geq 0$ every value at most 4 once. So

if $k > 4$: one zero; if $1 < k \leq 4$: two zeros; if $0 \leq k \leq 1$: one zero; if $k < 0$: two zeros.

Part II

5. A civil engineer has to design and build a fountain. The lot is bounded by a wall on the west and by a hedge on the south; the wall and the hedge are perpendicular to each other. On the lot there is a plum tree 7 m from both the wall and the hedge, a cherry tree 5 m from the wall and 3 m from the hedge, and a statue 18 m from the wall and 9 m from the hedge. The fountain has to be placed so that it is the same distance from the two trees, and it is twice as far from the statue as from the cherry tree.

- a) How far from the statue will the fountain be built?

(11 points)

- b) Ede would do the earthwork for the fountain in 10 hours, Béla in 12 hours. Béla starts at 7 in the morning, Ede joins him an hour later, they take a half-hour break once, and then they work together until the job is done. At what time do they finish?

(5 points)

Solution: a) Let the line of the wall be the y -axis and the line of the hedge the x -axis. The fountain is $K(x, y)$, the plum tree $S(7, 7)$, the cherry tree $C(5, 3)$, and the statue $B(18, 9)$. Let t be the distance of the fountain from the cherry tree:

$$(1) \quad (x - 5)^2 + (y - 3)^2 = t^2$$

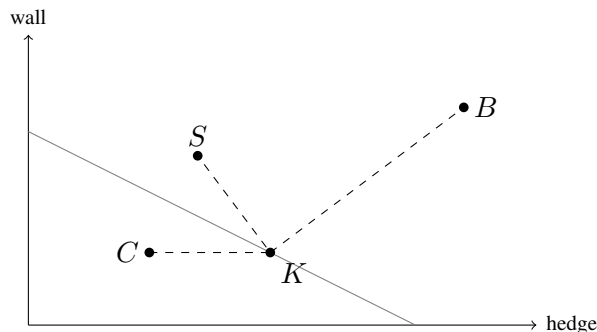
The fountain is the same distance from the two trees, so it is on the perpendicular bisector of CS . The midpoint is $F(6, 5)$, and $\overrightarrow{FS} = (1, 2)$ is a normal vector:

$$(2) \quad x + 2y = 16$$

The fountain is twice as far from the statue as from the cherry tree:

$$(3) \quad (x - 18)^2 + (y - 9)^2 = (2t)^2$$

From (1), (2), and (3) we get $3y^2 - 38y + 87 = 0$. The solutions are $x_1 = -\frac{10}{3}$, $y_1 = \frac{29}{3}$, which is outside the lot, and $x_2 = 10$, $y_2 = 3$, $t_2 = 5$. So the fountain can be built $2 \cdot 5 = 10$ m from the statue.



b) Let x be the time they work together (in hours). In the first hour Béla does $\frac{1}{12}$ of the work, so

$$\frac{1}{12} + \left(\frac{1}{12} + \frac{1}{10} \right) x = 1, \quad \text{which gives} \quad x = 5$$

So after 1 hour of Béla working alone, 5.5 hours pass until the end of the job (including the half-hour break). They finish at 1 : 30 p.m.

6. In an arithmetic sequence the sum of the reciprocals of the first and the fourth terms is 5.5. The first, second, and sixth terms of the sequence are three consecutive terms of a geometric sequence. Find the first term and the difference of the arithmetic sequence.

(16 points)

Solution: Let the first term be a and the difference d . Since a , $a + d$, $a + 5d$ form a geometric sequence, $a(a + 5d) = (a + d)^2$, and by the other condition,

$$\frac{1}{a} + \frac{1}{a + 3d} = 5.5$$

Rearranging the two equations:

$$\begin{aligned} d(3a - d) &= 0 \\ 5.5a^2 + 16.5ad - 3d - 2a &= 0 \end{aligned}$$

From the first equation, $d = 0$ or $d = 3a$.

If $d = 0$: $5.5a^2 - 2a = 0$, so $a = 0$ or $a = \frac{4}{11}$. $a = 0$ is impossible. So $a = \frac{4}{11}$, $d = 0$ is a solution (a constant sequence).

If $d = 3a$: $55a^2 - 11a = 0$, so $a = 0$ or $a = \frac{1}{5}$. So $a = \frac{1}{5}$, $d = \frac{3}{5}$ is a solution. (The terms $\frac{1}{5}$, $\frac{4}{5}$, $\frac{16}{5}$ form a geometric sequence with ratio 4.)

7. We roll two dice. Let X be the absolute value of the difference of the two numbers rolled.

a) What is the probability that X is a perfect square? (6 points)

b) Graph the distribution of the random variable X . (5 points)

c) Find the expected value of X . (5 points)

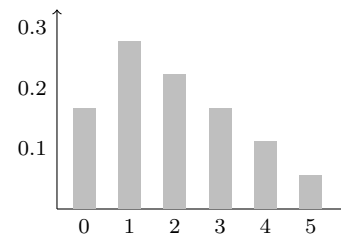
Solution: a) The table shows all possible cases; the favorable values (0, 1, 4) are underlined.

	1	2	3	4	5	6
1	<u>0</u>	<u>1</u>	2	3	<u>4</u>	5
2	<u>1</u>	<u>0</u>	<u>1</u>	2	3	<u>4</u>
3	2	<u>1</u>	<u>0</u>	<u>1</u>	2	3
4	3	2	<u>1</u>	<u>0</u>	<u>1</u>	2
5	<u>4</u>	3	2	<u>1</u>	<u>0</u>	<u>1</u>
6	5	<u>4</u>	3	2	<u>1</u>	<u>0</u>

There are 20 favorable cases out of 36, so $p = \frac{20}{36} = \frac{5}{9}$.

b)

X	0	1	2	3	4	5
frequency	6	10	8	6	4	2
$p(X)$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$



c)

$$E(X) = 0 \cdot \frac{6}{36} + 1 \cdot \frac{10}{36} + 2 \cdot \frac{8}{36} + 3 \cdot \frac{6}{36} + 4 \cdot \frac{4}{36} + 5 \cdot \frac{2}{36} = \frac{70}{36} = \frac{35}{18}$$

8. The sides of the triangle ABC are $a = 13$ cm, $b = 14$ cm, $c = 15$ cm.

a) Find the length of the median s_a from the vertex A and of the angle bisector f_c to the side c . (8 points)

b) Find the radii of the inscribed and the circumscribed circles of the triangle. (8 points)

Solution: a) By the Law of Cosines in the triangle ABC , $15^2 = 13^2 + 14^2 - 2 \cdot 13 \cdot 14 \cdot \cos \gamma$, so $\cos \gamma = \frac{5}{13}$, $\gamma \approx 67.38^\circ$. Let F be the midpoint of the side BC . By the Law of Cosines in the triangle AFC :

$$s_a^2 = 6.5^2 + 14^2 - 2 \cdot 6.5 \cdot 14 \cdot \frac{5}{13} = 168.25, \quad \text{so} \quad s_a = \sqrt{168.25} \approx 12.97$$

The bisector from C meets the opposite side at D ; it divides the side in the ratio of the adjacent sides: $\frac{BD}{15 - BD} = \frac{13}{14}$, so $BD = \frac{65}{9}$. By the Law of Cosines, $\cos \beta = \frac{33}{65}$, and in the triangle BCD :

$$f_c^2 = 13^2 + \left(\frac{65}{9}\right)^2 - 2 \cdot 13 \cdot \frac{65}{9} \cdot \frac{33}{65} = \frac{10192}{81}, \quad \text{so} \quad f_c \approx 11.22$$

The median s_a is about 12.97 cm, and the angle bisector f_c is about 11.22 cm.

b) By Heron's formula, the area is $t = \sqrt{21 \cdot 8 \cdot 7 \cdot 6} = 84$. The inradius is $\rho = \frac{t}{s} = \frac{84}{21} = 4$ cm, and the circumradius is

$$r = \frac{abc}{4t} = \frac{13 \cdot 14 \cdot 15}{4 \cdot 84} = \frac{65}{8} = 8.125 \text{ cm}$$

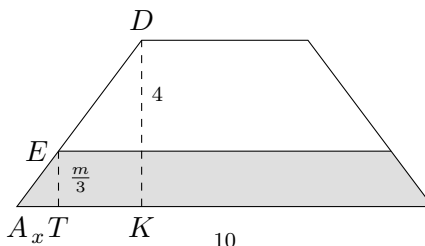
9. A container in the shape of a right prism whose base is an isosceles trapezoid holds some water. The parallel sides of the trapezoid are 4 cm and 10 cm, its legs are 5 cm, and the length of the prism is 11.2 cm. If we turn the container from its trapezoid-shaped base onto its largest lateral face, the height of the water becomes one third of what it was. Find the volume of the water in the container.

(16 points)

Solution: The height of the trapezoid is $\sqrt{5^2 - 3^2} = 4$ cm. If the container stands on its trapezoid base and the water is m cm high, the volume of the water is

$$V = \frac{10 + 4}{2} \cdot 4 \cdot m = 28m$$

The largest lateral face is the 10 by 11.2 rectangle. When the container lies on this face, the water fills a trapezoid-shaped part of the cross section, of height $\frac{m}{3}$.



The triangles KDA and TEA are similar, so $\frac{x}{3} = \frac{m/3}{4}$, that is, $x = \frac{m}{4}$. The volume of the water is unchanged:

$$\frac{10 + (10 - 2 \cdot \frac{m}{4})}{2} \cdot \frac{m}{3} \cdot 11.2 = 28m$$

This gives $11.2m^2 - 112m = 0$, so $m = 0$ or $m = 10$. For $m = 0$ there is no water in the container. For $m = 10$ (which fits, since $10 < 11.2$ and $\frac{10}{3} < 4$), the container holds $28 \cdot 10 = 280$ cm³ of water.