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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. We poured together 5 kg of 15 percent, 8 kg of 20 percent, and 12 kg of 40 percent (by mass) sugar solution. The solvent was water in each case. Since then, 40 dkg (0.4 kg) of sugar fell into it, 12 percent of the water evaporated, and the mixture is uniform.

- What is the concentration of the solution now (percent by mass)?
- What fraction of the solution has to be replaced by water if we want to get a 10 percent solution?

(11 points)

Solution: a) The 25 kg of solution poured together contains $5 \cdot 0.15 + 8 \cdot 0.2 + 12 \cdot 0.4 = 7.15$ kg of sugar and 17.85 kg of water. After the sugar fell in, there is 7.55 kg of sugar, and after the evaporation $17.85 \cdot 0.88 = 15.708$ kg of water. The concentration is

$$\frac{7.55}{7.55 + 15.708} \cdot 100 \approx 32.46\%$$

- b) The 23.258 kg of solution contains 7.55 kg of sugar. After the replacement the mass of the solution does not change, so it should contain 2.3258 kg of sugar. If we replace the fraction f of the (uniform) solution with water, then $(1 - f) \cdot 7.55 = 2.3258$, so

$$f = 1 - \frac{2.3258}{7.55} \approx 0.692$$

About 0.692 of the solution has to be replaced by water.

2. The following table shows the populations of the continents in 1995.

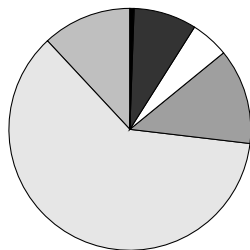
Continent	Population (millions)
Europe	682
Asia	3498
Africa	730
North America	293
Central and South America	481
Australia and Oceania	32

- Find the mean and the standard deviation of the data.
- Make a diagram of the distribution of the Earth's population by continent.

(12 points)

Solution: a) The mean population of the continents is $\frac{5716}{6} \approx 952.7$ million, and the standard deviation is about 1162.4 million.

- b) The distribution can be shown well, for example, by a pie chart. The central angles of the sectors are proportional to the populations. (Of course, any well-made diagram is acceptable.)



	Continent	Central angle
	Europe	42.95°
	Asia	220.31°
	Africa	45.98°
	North America	18.45°
	Central and South America	30.29°
	Australia and Oceania	2.02°

3. a) We know that p and $10p - 1$ are positive primes. Can $10p + 1$ be a prime?

b) Péter and Pál are both very clever. Once Péter said to Pál:

– Three people just visited me. The sum of their ages is 4 more than your age, the product of their ages is twice the square of 35, and each of them is older than four.

– I do not know the answer, said Pál after a short thought.

– One of my visitors is older than me, Péter continued the conversation.

– Then I know the answer, said Pál.

What is the difference between the ages of Péter and Pál?

(14 points)

Solution: a) If $p = 3$, then $10p - 1 = 29$ is a prime, and $10p + 1 = 31$ is also a prime.

If $p \neq 3$ is a prime, then $10p - 1$ is a prime that is at least 19, so it is not divisible by 3. Clearly $10p$ is not divisible by 3 either, so $10p + 1$ is divisible by 3 (of three consecutive integers, exactly one is divisible by 3). Since $10p + 1$ is greater than 3, it is not a prime.

So $10p + 1$ is a prime only for $p = 3$.

b) Pál computed that twice the square of 35 is $2450 = 2 \cdot 5^2 \cdot 7^2$, and it can be written as a product of three factors, each greater than 4, in the following ways:

factors			sum
5	5	98	108
5	7	70	82
5	10	49	64
5	14	35	54
7	7	50	64
7	10	35	52
7	14	25	46

Pál knows his own age, and he could not decide, so he could not choose between the two bold cases: Pál is $64 - 4 = 60$ years old. When Péter said that one of his visitors is older than him, Pál could decide. So Péter is 49 years old: if he were younger, both cases would remain possible, and if he were older, neither case would be possible. (The visitors were 7, 7, and 50 years old.) So Pál is 11 years older than Péter.

4. a) We are given 2007 points in the plane, no three of which are on a line. Three of these points are green. Consider all triangles determined by the 2007 points. How many of them have a green vertex?
- b) Prove that any two neighboring terms of the Fibonacci sequence are relatively prime. (The rule of the Fibonacci sequence: $a_1 = 1, a_2 = 1, a_n = a_{n-1} + a_{n-2}$ for every $n > 2$.)

(14 points)

Solution: a) The points determine $\binom{2007}{3}$ triangles in all, and $\binom{2004}{3}$ of them have no green vertex. So

$$\binom{2007}{3} - \binom{2004}{3} = 6\,027\,031$$

triangles have a green vertex.

b) Indirect proof: suppose that two neighboring terms a_k and a_{k-1} are not relatively prime, that is, there is an integer $m > 1$ with $m \mid a_k$ and $m \mid a_{k-1}$. Then $m \mid (a_k - a_{k-1}) = a_{k-2}$. In the same way, $m \mid a_{k-1}$ and $m \mid a_{k-2}$ give $m \mid a_{k-3}$. After finitely many steps we get $m \mid a_2 = 1$, which is a contradiction. So the statement is true.

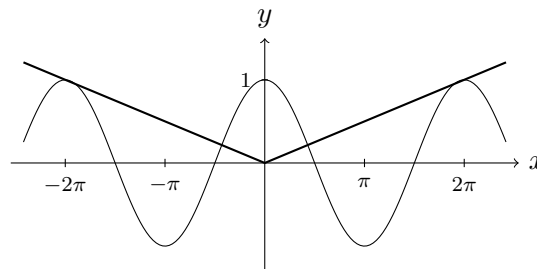
Part II

5. a) Solve the equation $2x^3 - 3x^2 + 3x - 1 = 0$.
- b) How many solutions does the equation $\cos x = \frac{|x|}{2\pi}$ have?

(16 points)

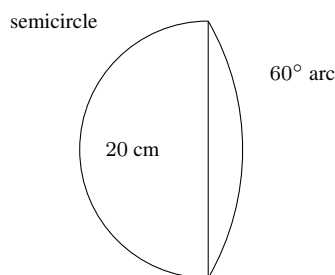
Solution: a) We rewrite the equation as $x^3 - 3x^2 + 3x - 1 = -x^3$, that is, $(x-1)^3 = -x^3 = (-x)^3$. Since the cube function is one-to-one, $x-1 = -x$, so $x = \frac{1}{2}$. This is the only solution.

b) We graph both sides in the same coordinate system.



For $x > 2\pi$ the right side is greater than 1, so there are no solutions there. For $0 \leq x \leq 2\pi$: there is one solution in $(0, \frac{\pi}{2})$; there is none in $[\frac{\pi}{2}, \frac{3\pi}{2}]$, where $\cos x \leq 0 < \frac{x}{2\pi}$; $x = 2\pi$ is a solution, and since there the slope of $\cos x$ is 0 while the line rises, $\cos x$ is above the line just before 2π , but below it at $\frac{3\pi}{2}$, so there is one more solution in $(\frac{3\pi}{2}, 2\pi)$. Both sides are even functions, and $x = 0$ is not a solution. So the equation has exactly 6 solutions.

6. The figure shows the cross section of wooden fence posts. Each of the 80 fence posts is 3 meters long, and 1 meter of each will be in the ground. This part of the fence posts is soaked with a special liquid to make them last longer.



- a) How much liquid is needed if 1 m^3 of wood “absorbs” 2.5 liters of liquid?
 b) How much paint is needed to paint the fence posts after they are set up, if 1 kg of the paint covers 4 m^2 ?

(16 points)

Solution: The cross section consists of a semicircle of radius $r = 10 \text{ cm}$ and a circular segment of radius $R = 20 \text{ cm}$ with central angle 60° (its chord is 20 cm).

- a) The area of the semicircle is $T_1 = \frac{r^2\pi}{2} \approx 157.08 \text{ cm}^2$, and the area of the segment is

$$T_2 = \frac{1}{6}R^2\pi - \frac{R^2 \sin 60^\circ}{2} \approx 36.23 \text{ cm}^2$$

The area of the cross section is $T = T_1 + T_2 \approx 193.31 \text{ cm}^2$. The part of one post in the ground is 1 m long, so its volume is $193.31 \cdot 100 \approx 19\,331 \text{ cm}^3$, that is, about 0.0193 m^3 . The liquid needed is $80 \cdot 0.0193 \cdot 2.5 \approx 3.9$ liters.

- b) The arc of the semicircle is $k_1 = r\pi \approx 31.42 \text{ cm}$, and the arc of the segment is $k_2 = \frac{1}{6} \cdot 2R\pi \approx 20.94 \text{ cm}$, so the perimeter of the cross section is $k \approx 52.36 \text{ cm}$. We paint the top and the $m' = 200 \text{ cm}$ long part above the ground:

$$A = 80 \cdot T + 80 \cdot k \cdot m' \approx 853\,225 \text{ cm}^2 \approx 85.3 \text{ m}^2$$

The paint needed is $\frac{A}{4} \approx 21.33 \text{ kg}$.

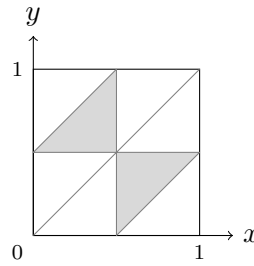
7. Shipwrecked people on an uninhabited island without vegetation plan to build a new, smaller boat from the pieces of their original sailboat, which ran aground in the storm. The storm broke the mast at random into three pieces. If a piece at least 20 m long remained of the original 40 m long mast, the boat can be built. What is the probability that when they swim back to the wreck, they find such a piece?

(16 points)

Solution: Taking 40 m as the unit, the question is: if we choose two points of the unit segment at random, dividing it into three parts, what is the probability that one of the parts is at least $\frac{1}{2}$ long? Let the two points be x and y ; then (x, y) is a random point of the unit square. We compute the probability p' of the complement: none of the parts is longer than $\frac{1}{2}$.

If $x < y$: $x < \frac{1}{2}$, $y - x < \frac{1}{2}$, and $1 - y < \frac{1}{2}$, that is, $y < x + \frac{1}{2}$ and $y > \frac{1}{2}$.

If $x > y$: $y < \frac{1}{2}$, $x - y < \frac{1}{2}$, and $1 - x < \frac{1}{2}$, that is, $y > x - \frac{1}{2}$ and $x > \frac{1}{2}$.



The points satisfying these conditions form the two shaded triangles, so $p' = \frac{1}{4}$. The probability that the shipwrecked people find a suitable piece is $p = 1 - p' = 0.75$.

8. A company ordered 100 closed containers in the shape of a cylinder, each with an inner volume of 1 m^3 , made of special sheet metal. The customer specified only the volume and the shape. What should the height of the cylinder be if the company wants to use as little material as possible?

(16 points)

Solution: We find the height for which the surface area of a cylinder of volume 1 m^3 is the smallest. With the usual notation, $V = r^2\pi m = 1$, so $r^2 = \frac{1}{\pi m}$, and the surface area is

$$A = 2r^2\pi + 2r\pi m = \frac{2}{m} + 2\sqrt{\frac{1}{\pi m}} \cdot \pi m = \frac{2}{m} + 2\sqrt{\pi m}$$

We look for the minimum of $A(m) = \frac{2}{m} + 2\sqrt{\pi m}$ ($m > 0$). The derivative is

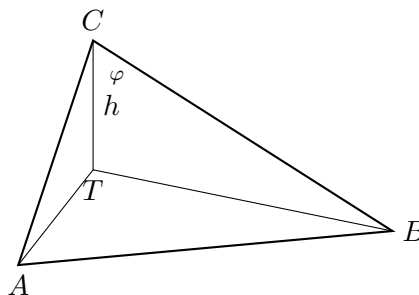
$$A'(m) = -\frac{2}{m^2} + \frac{\sqrt{\pi}}{\sqrt{m}} = 0, \quad \text{so} \quad \sqrt{\pi} \cdot \sqrt{m^3} = 2, \quad m = \sqrt[3]{\frac{4}{\pi}} \approx 1.08$$

A' changes from negative to positive here, so the surface area is minimal. The height of the cylinder should be about 1.08 m. (Then $m = 2r$: the height equals the diameter.)

9. A commander examines, from the top of a 480 m high hill, the position of his own artillery, stationed on flat terrain along the seashore, and of an enemy infantry corps, also on flat terrain along the seashore. The apparent distance of the two corps (the angle between the lines of sight) is 71.2° . The commander sees the artillery at an angle of depression of $6^\circ 30'$, and the infantry at an angle of depression of 8° . Will the commander, who is good at mathematics, give the order to fire if he knows that the maximum range of his artillery's cannons is 4 km?

(16 points)

Solution: Let the infantry be at A , the artillery at B , and the commander at C , at height $h = 480 \text{ m}$ above the point T . The data: $\alpha = 8^\circ$, $\beta = 6^\circ 30'$, $\varphi = \angle ACB = 71.2^\circ$.



From the right triangles ATC and BTC :

$$AC = \frac{h}{\sin \alpha} \approx 3448.94 \text{ m}, \quad BC = \frac{h}{\sin \beta} \approx 4240.16 \text{ m}$$

By the Law of Cosines, $AB^2 = AC^2 + BC^2 - 2 \cdot AC \cdot BC \cdot \cos \varphi$, so $AB \approx 4522 \text{ m}$. This is more than 4 km, so the commander, who is good at mathematics, will not give the order to fire.