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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. Solve the following equations over the real numbers.

a) $\sqrt{\log_2^2 x - 4 \log_2 x + 4} = \log_2 \frac{4}{x}$ (6 points)

b) $\sqrt{5 - \cos^2 x - 4 \sin x} = 2 - \sin x$ (6 points)

2. A bartender calculated that if he added 12 liters of water to his 60% alcohol, he could sell it as 42% alcohol.

a) How many liters of the original 60% alcohol were there? (6 points)

b) Hoping for a bigger profit, the bartender poured 16 liters of water instead of 12 into the original alcohol. He knew from experience that customers do not notice a deviation of $\pm 2\%$ from 42% alcohol without an instrument, but they do notice a larger deviation. Will the customers notice the cheating? (6 points)

3. In a survey, 52 families were asked about the number of children in the family and their genders. The table shows the results.

		number of boys					
		0	1	2	3	4	5
number of girls	0	6	1	2	2	1	0
	1	2	3	2	1	2	1
	2	4	3	2	1	3	1
	3	3	2	2	1	1	0
	4	1	1	1	1	0	0
	5	0	1	0	1	0	0

(For example, there were 6 families with no children, and 3 families with one boy and two girls.)

a) What is the average number of children in a family? (5 points)

b) How many boys and how many girls are there altogether in the surveyed families? (4 points)

c) We choose 2 of the surveyed families at random. What is the probability that each of the two families has at least 3 children? (4 points)

4. A three-volume literary encyclopedia can also be bought volume by volume. A bookstore recorded for one week how many copies of each volume were sold on each day. The table shows the results.

	Mon	Tue	Wed	Thu	Fri	Sat
Volume I	2	1	4	2	4	2
Volume II	3	2	2	1	0	2
Volume III	5	3	4	6	3	3

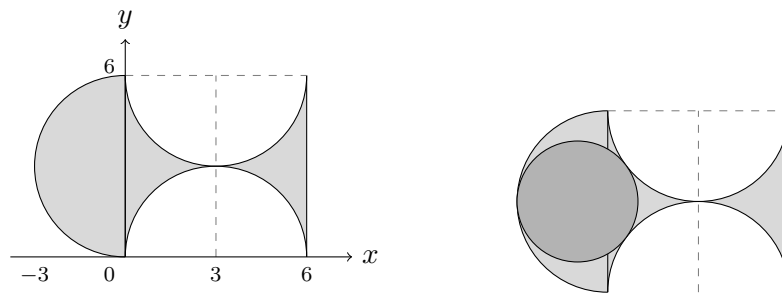
Four customers bought all three volumes (nobody bought more than one copy of a volume). Of those who bought the first volume, 6 did not buy the second volume, and also 6 did not buy the third volume.

- a) If there was no customer who bought only the second volume, how many bought only the third volume? (6 points)
- b) How many customers bought volumes of the encyclopedia during this week? (3 points)
- c) Each volume costs 3600 forints. A customer who bought two volumes got a 10% discount on one of them. A customer who bought all three volumes got a 10% discount on one volume and a 20% discount on another. What was the store's revenue from these volumes this week? (5 points)

Part II

5. The figure shows the axial cross section of a decorative element in the shape of a solid of revolution, placed in a coordinate system. The curves bounding the cross section are

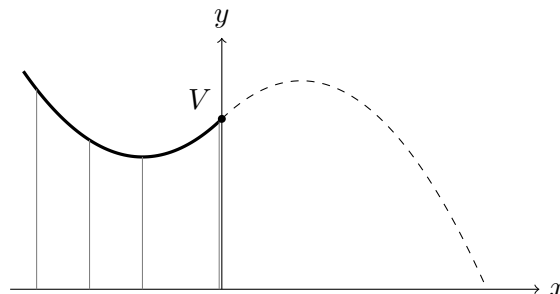
$$x^2 + (y - 3)^2 = 9 \quad (x \leq 0), \quad (x - 3)^2 + y^2 = 9 \quad (y \geq 0), \quad (x - 3)^2 + (y - 6)^2 = 9 \quad (y \leq 6)$$



- a) What is the area of the cross section? (6 points)
 - b) To make the solid more stable, a ball is placed into it that cannot move. (In the cross section, this is a great circle that touches two of the semicircles from outside and one from inside; see the figure on the right.) What is the radius of this ball? (10 points)
6. The figure shows the landing slope of a ski jump in a coordinate system. The slope fits the graph of the function

$$f(x) = \frac{4}{250}x^2 + \frac{24}{25}x + \frac{322}{5}$$

where the x -axis is at an altitude of 1000 m above sea level, and the end point V of the ramp is on the y -axis. A skier leaving the point V flies along a parabola that is the reflection of the parabola of the ramp in the point V .

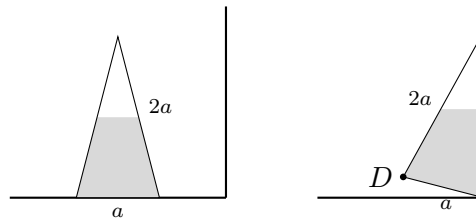


- a) To what height above sea level does the skier fly up? (4 points)
- b) How far from the support column at the end point V of the ramp does the skier reach the altitude of 1000 m above sea level? (6 points)
- c) Find the equation of the tangent line of the skier's flight path at the moment he leaves the ramp. (6 points)

7. The base of a right prism is an isosceles triangle whose legs are twice its base. The prism lies on the floor next to a wall, on the lateral face containing the base of the triangle, and in this position it is filled with water to half of its height (see the figure on the left). The thickness of the walls can be neglected.

a) How high does the water stand in the solid if we rotate it and lean it against the wall (see the figure on the right)? (8 points)

b) There is a plug at the vertex D of the solid. After leaning the prism against the wall as described, we pulled out the plug, and the water flows out of the solid at an average rate of 12 liters per minute. How long does the water flow if $a = 1$ m and $h = 4$ m? (8 points)



(cross sections of the prism; h is the length of the prism)

8. a) The first term of an arithmetic sequence is $\overline{ab}$, its second term is $\overline{ba}$ (two-digit numbers), and its third term is the three-digit number $\overline{acb}$. What is the difference of this sequence? (9 points)

b) Let a_n and b_n be non-constant arithmetic sequences of positive integers. Prove that $a_{b_n} - b_{a_n}$ is a constant independent of n . (7 points)

9. a) Find the domain of the following expression.

$$\sqrt{x^2 - 15x + 36} + \sqrt{-x^2 + 19x - 84} + \log_{\frac{x}{2}} 36$$

(6 points)

- b) Which real numbers x, y, z satisfy the following equation?

$$\sqrt{x^2 - 15x + 36} + \sqrt{-x^2 + 19x - 84} + \log_{\frac{x}{2}} 36 = 2 \cos 3z - y^2 + 4y - 4$$

(10 points)