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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. Solve the following equations over the real numbers.

a) $\sqrt{\log_2^2 x - 4 \log_2 x + 4} = \log_2 \frac{4}{x}$ (6 points)

b) $\sqrt{5 - \cos^2 x - 4 \sin x} = 2 - \sin x$ (6 points)

Solution: a) Because of the logarithms, $x > 0$. The right side is $\log_2 \frac{4}{x} = 2 - \log_2 x$. With the new unknown $a = \log_2 x$: $\sqrt{a^2 - 4a + 4} = 2 - a$. Under the square root there is a perfect square, so the equation is $|a - 2| = 2 - a$, which holds exactly when $a \leq 2$. So $\log_2 x \leq 2 = \log_2 4$, and since $\log_2 x$ is increasing, $x \leq 4$. The solutions are the real numbers $0 < x \leq 4$.

b) Using $\cos^2 x = 1 - \sin^2 x$, the equation is $\sqrt{\sin^2 x - 4 \sin x + 4} = 2 - \sin x$, that is, $|\sin x - 2| = 2 - \sin x$. This holds when $\sin x \leq 2$, which is true for every x . So every real number is a solution.

2. A bartender calculated that if he added 12 liters of water to his 60% alcohol, he could sell it as 42% alcohol.

a) How many liters of the original 60% alcohol were there? (6 points)

b) Hoping for a bigger profit, the bartender poured 16 liters of water instead of 12 into the original alcohol. He knew from experience that customers do not notice a deviation of $\pm 2\%$ from 42% alcohol without an instrument, but they do notice a larger deviation. Will the customers notice the cheating? (6 points)

Solution: a) Let the original amount be x liters; it contains $0.6x$ liters of pure alcohol. After adding 12 liters of water:

$$\frac{0.6x}{x + 12} \cdot 100 = 42, \quad \text{that is,} \quad 60x = 42x + 504, \quad x = 28$$

There were 28 liters of 60% alcohol.

b) With 16 liters of water the concentration is

$$\frac{28 \cdot 0.6}{28 + 16} \cdot 100 \approx 38.18\%$$

This differs from 42% by more than 2%, so the customers will notice the cheating.

3. In a survey, 52 families were asked about the number of children in the family and their genders. The table shows the results.

		number of boys						
		0	1	2	3	4	5	
number of girls	0	6	1	2	2	1	0	
	1	2	3	2	1	2	1	
	2	4	3	2	1	3	1	
	3	3	2	2	1	1	0	
	4	1	1	1	1	0	0	
	5	0	1	0	1	0	0	

(For example, there were 6 families with no children, and 3 families with one boy and two girls.)

- a) What is the average number of children in a family? (5 points)
- b) How many boys and how many girls are there altogether in the surveyed families? (4 points)
- c) We choose 2 of the surveyed families at random. What is the probability that each of the two families has at least 3 children? (4 points)

Solution: a) From the table, the number of families with 0, 1, 2, ... children:

number of children	0	1	2	3	4	5	6	7	8
number of families	6	3	9	10	7	6	7	3	1

The average number of children is

$$\frac{1 \cdot 3 + 2 \cdot 9 + 3 \cdot 10 + 4 \cdot 7 + 5 \cdot 6 + 6 \cdot 7 + 7 \cdot 3 + 8 \cdot 1}{52} = \frac{180}{52} \approx 3.46$$

- b) The number of families with 1 boy is 11 (11 boys); with 2 boys, 9 (18 boys); with 3 boys, 7 (21 boys); with 4 boys, 7 (28 boys); with 5 boys, 2 (10 boys). So there are $11 + 18 + 21 + 28 + 10 = 88$ boys, and $180 - 88 = 92$ girls.
- c) Two of the 52 families can be chosen in $\binom{52}{2}$ ways. From the table, 34 families have at least 3 children, and two of them can be chosen in $\binom{34}{2}$ ways. So the probability is

$$\frac{\binom{34}{2}}{\binom{52}{2}} = \frac{34 \cdot 33}{52 \cdot 51} = \frac{1122}{2652} \approx 0.423$$

4. A three-volume literary encyclopedia can also be bought volume by volume. A bookstore recorded for one week how many copies of each volume were sold on each day. The table shows the results.

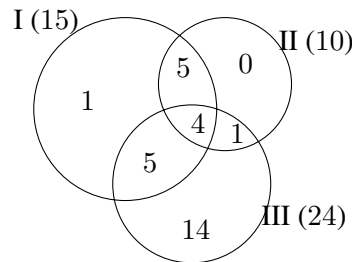
	Mon	Tue	Wed	Thu	Fri	Sat
Volume I	2	1	4	2	4	2
Volume II	3	2	2	1	0	2
Volume III	5	3	4	6	3	3

Four customers bought all three volumes (nobody bought more than one copy of a volume). Of those who bought the first volume, 6 did not buy the second volume, and also 6 did not buy the third volume.

- a) If there was no customer who bought only the second volume, how many bought only the third volume? (6 points)

- b) How many customers bought volumes of the encyclopedia during this week? (3 points)
- c) Each volume costs 3600 forints. A customer who bought two volumes got a 10% discount on one of them. A customer who bought all three volumes got a 10% discount on one volume and a 20% discount on another. What was the store's revenue from these volumes this week? (5 points)

Solution: a) From the table, 15 customers bought Volume I, 10 bought Volume II, and 24 bought Volume III. In the Venn diagram, let a, t, s be the numbers of customers who bought only Volume I, II, III; let y, x, u be the numbers who bought exactly Volumes I and II, I and III, II and III.



By the conditions, $a + x = 6$, $a + y = 6$, and $t = 0$. Volume I: $a + x + y + 4 = 15$, so $y = 5$, $a = 1$, $x = 5$. Volume II: $5 + 4 + u + 0 = 10$, so $u = 1$. Volume III: $5 + 4 + 1 + s = 24$, so $s = 14$. So 14 customers bought only the third volume.

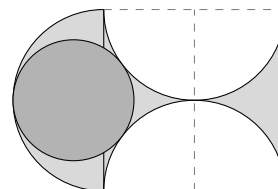
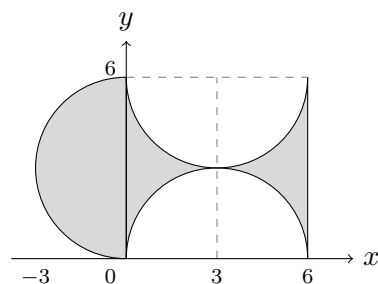
- b) $1 + 5 + 4 + 5 + 1 + 0 + 14 = 30$ customers bought volumes this week.
- c) Each of the 30 customers paid the full price of one volume. The 15 customers who bought at least two volumes also paid 90% of the price for a second volume, and the 4 customers who bought all three also paid 80% of the price for a third volume:

$$30 \cdot 3600 + 15 \cdot 3600 \cdot 0.9 + 4 \cdot 3600 \cdot 0.8 = 108\,000 + 48\,600 + 11\,520 = 168\,120 \text{ forints}$$

Part II

5. The figure shows the axial cross section of a decorative element in the shape of a solid of revolution, placed in a coordinate system. The curves bounding the cross section are

$$x^2 + (y - 3)^2 = 9 \quad (x \leq 0), \quad (x - 3)^2 + y^2 = 9 \quad (y \geq 0), \quad (x - 3)^2 + (y - 6)^2 = 9 \quad (y \leq 6)$$

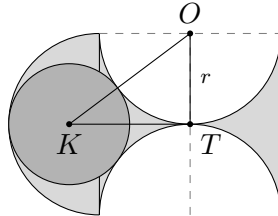


- a) What is the area of the cross section? (6 points)
- b) To make the solid more stable, a ball is placed into it that cannot move. (In the cross section, this is a great circle that touches two of the semicircles from outside and one from inside; see the figure on the right.) What is the radius of this ball? (10 points)

Solution: a) The cross section consists of a semicircle of radius $r = 3$ and a square of side $2r$ from which a full circle of radius r (two semicircles) is removed. Its area is

$$\frac{r^2\pi}{2} + 4r^2 - r^2\pi = 4r^2 - \frac{r^2\pi}{2} = 36 - \frac{9\pi}{2} \approx 21.86$$

b) Let x be the radius of the ball and K its center; by symmetry K is on the line $y = 3$. Let $O(3, 6)$ be the center of the upper semicircle and $T(3, 3)$.



The ball touches the left semicircle (center $(0, 3)$, radius 3) from inside, so $K = (x - 3, 3)$, and $TK = 2r - x$. It touches the upper semicircle from outside, so $OK = r + x$, and $OT = r$. By the Pythagorean theorem in the right triangle KTO :

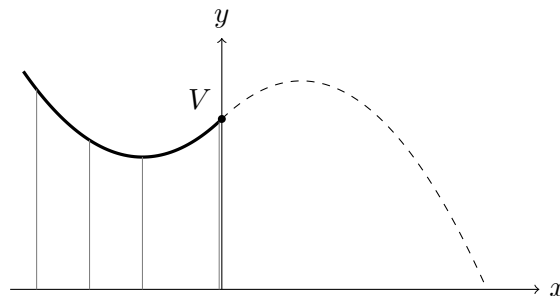
$$(r + x)^2 = r^2 + (2r - x)^2, \quad \text{that is,} \quad 6rx = 4r^2$$

So $x = \frac{2}{3}r = 2$. The radius of the ball is 2.

6. The figure shows the landing slope of a ski jump in a coordinate system. The slope fits the graph of the function

$$f(x) = \frac{4}{250}x^2 + \frac{24}{25}x + \frac{322}{5}$$

where the x -axis is at an altitude of 1000 m above sea level, and the end point V of the ramp is on the y -axis. A skier leaving the point V flies along a parabola that is the reflection of the parabola of the ramp in the point V .



- To what height above sea level does the skier fly up? (4 points)
- How far from the support column at the end point V of the ramp does the skier reach the altitude of 1000 m above sea level? (6 points)
- Find the equation of the tangent line of the skier's flight path at the moment he leaves the ramp. (6 points)

Solution: a) Completing the square:

$$\frac{4}{250}x^2 + \frac{24}{25}x + \frac{322}{5} = \frac{4}{250} [x^2 + 60x + 4025] = \frac{4}{250} (x + 30)^2 + 50$$

The ramp has its minimum 50 at $x = -30$, and it meets the y -axis at $V(0, 64.4)$. The reflection of the minimum point $(-30, 50)$ in V is $(30, 78.8)$, so the skier flies along the curve

$$y = -\frac{4}{250}(x - 30)^2 + 78.8$$

He flies up to 1078.8 m above sea level.

b) We need the positive zero of the flight path: $(x - 30)^2 = \frac{250 \cdot 78.8}{4} = 4925$, so $x \approx 100.2$. The skier reaches the altitude of 1000 m about 100.2 m from the support column at V .

c) The tangent line passes through $V(0, 64.4)$, and its slope is the derivative of the flight path at $x = 0$:

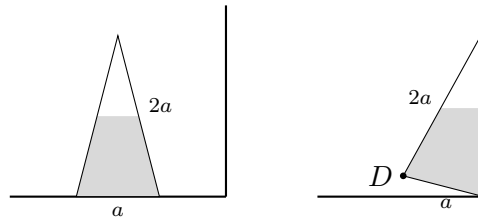
$$y' = -\frac{8}{250}(x - 30), \quad y'(0) = \frac{24}{25}$$

The tangent line is $y = \frac{24}{25}x + 64.4$.

7. The base of a right prism is an isosceles triangle whose legs are twice its base. The prism lies on the floor next to a wall, on the lateral face containing the base of the triangle, and in this position it is filled with water to half of its height (see the figure on the left). The thickness of the walls can be neglected.

a) How high does the water stand in the solid if we rotate it and lean it against the wall (see the figure on the right)? (8 points)

b) There is a plug at the vertex D of the solid. After leaning the prism against the wall as described, we pulled out the plug, and the water flows out of the solid at an average rate of 12 liters per minute. How long does the water flow if $a = 1$ m and $h = 4$ m? (8 points)



(cross sections of the prism; h is the length of the prism)

Solution: We compute with $a = 1$; every length is proportional to a . The base triangle ABC has $AB = 1$, $AC = BC = 2$; its height is $m = \sqrt{4 - \frac{1}{4}} = \frac{\sqrt{15}}{2}$, and its area is $T = \frac{\sqrt{15}}{4}$.

a) In the original position, the empty part of the cross section is an isosceles triangle similar to the base triangle with ratio 1 : 2, so its area is $\frac{T}{4} = \frac{\sqrt{15}}{16}$.

In the leaning position, the leg BC is vertical along the wall, with B on the floor. The water surface is horizontal, so the empty part is a right triangle with the angle φ of the apex C ; let its legs be x (on BC) and y (horizontal). The amount of water does not change, so

$$\frac{xy}{2} = \frac{\sqrt{15}}{16}, \quad \text{and} \quad \tan \varphi = \frac{y}{x}$$

By the Law of Cosines in the base triangle, $1 = 4 + 4 - 8 \cos \varphi$, so $\cos \varphi = \frac{7}{8}$, $\sin \varphi = \frac{\sqrt{15}}{8}$, and $\tan \varphi = \frac{\sqrt{15}}{7}$. Then

$y = \frac{\sqrt{15}}{7}x$, and

$$\frac{x^2}{2} \cdot \frac{\sqrt{15}}{7} = \frac{\sqrt{15}}{16}, \quad \text{so} \quad x^2 = \frac{7}{8}, \quad x = \sqrt{\frac{7}{8}}$$

So the water stands $2 - \sqrt{\frac{7}{8}} \approx 1.065$ units high; in general, about $1.065a$ high (1.065 m for $a = 1$ m).

b) Let $D = A$ be the vertex with the plug, and let T be the foot of the altitude from C and E the foot of the perpendicular from A to BC . The water flows out until its level drops to the height of A ; the remaining water is the triangle ABE . The triangles ABE and CBT are similar with ratio $1 : 2$ (both are right triangles with the angle at B), so the area of ABE is one fourth of the area of CBT , that is, $\frac{1}{8}$ of the area of the base triangle. Since the empty part is $\frac{1}{4}$ of the base triangle, the water that flows out has a cross section of $1 - \frac{1}{4} - \frac{1}{8} = \frac{5}{8}$ of the base triangle, a right trapezoid:

$$T_{\text{trapezoid}} = \frac{\sqrt{15}}{4} \cdot \frac{5}{8} = \frac{5\sqrt{15}}{32} \text{ m}^2, \quad V = \frac{5\sqrt{15}}{32} \cdot 4 \approx 2.4206 \text{ m}^3 = 2420.6 \text{ liters}$$

At 12 liters per minute, the water flows for $\frac{2420.6}{12} \approx 201.7$ minutes.

8. a) The first term of an arithmetic sequence is $\overline{ab}$, its second term is $\overline{ba}$ (two-digit numbers), and its third term is the three-digit number $\overline{acb}$. What is the difference of this sequence? (9 points)

b) Let a_n and b_n be non-constant arithmetic sequences of positive integers. Prove that $a_{b_n} - b_{a_n}$ is a constant independent of n . (7 points)

Solution: a) Since the three numbers are consecutive terms of an arithmetic sequence,

$$\frac{\overline{ab} + \overline{acb}}{2} = \overline{ba}$$

Only $a = 1$ is possible: for $a \geq 2$ the numerator on the left side would be greater than 200, and the right side could not be a two-digit number. So $\overline{1b} + \overline{1cb} = 2 \cdot \overline{b1}$, that is, $10 + b + 100 + 10c + b = 20b + 2$, so $108 = 18b - 10c$. Since 108 and $18b$ are divisible by 9, $10c$ must also be divisible by 9, so $c = 0$ (then $b = 6$) or $c = 9$ (then $b = 11$, which is not a digit). The first three terms of the sequence are 16, 61, 106, and the difference is $d = 45$.

b) Let the first term of a_n be a and its difference d_1 , and let the first term of b_n be b and its difference d_2 . Then

$$\begin{aligned} a_{b_n} &= a + (b_n - 1)d_1 = a + [b + (n - 1)d_2 - 1]d_1 = a + bd_1 + nd_2d_1 - d_2d_1 - d_1 \\ b_{a_n} &= b + (a_n - 1)d_2 = b + [a + (n - 1)d_1 - 1]d_2 = b + ad_2 + nd_1d_2 - d_1d_2 - d_2 \end{aligned}$$

So $a_{b_n} - b_{a_n} = a - b + bd_1 - ad_2 - d_1 + d_2$, which is indeed a constant independent of n .

9. a) Find the domain of the following expression.

$$\sqrt{x^2 - 15x + 36} + \sqrt{-x^2 + 19x - 84} + \log_{\frac{x}{2}} 36$$

(6 points)

b) Which real numbers x, y, z satisfy the following equation?

$$\sqrt{x^2 - 15x + 36} + \sqrt{-x^2 + 19x - 84} + \log_{\frac{x}{2}} 36 = 2 \cos 3z - y^2 + 4y - 4$$

(10 points)

Solution: a) The conditions are

$$x^2 - 15x + 36 \geq 0, \quad -x^2 + 19x - 84 \geq 0, \quad x > 0, \quad x \neq 2$$

The zeros of $x^2 - 15x + 36$ are 3 and 12, so the first inequality holds for $x \leq 3$ or $x \geq 12$. The zeros of $-x^2 + 19x - 84$ are 7 and 12, so the second inequality holds for $7 \leq x \leq 12$. Together with the other two conditions, the expression is defined only for $x = 12$.

b) Substituting $x = 12$:

$$\log_6 36 = 2 \cos 3z - (y^2 - 4y + 4), \quad \text{that is,} \quad 2 = 2 \cos 3z - (y - 2)^2$$

The right side is at most 2, with equality only if $\cos 3z = 1$ and $(y - 2)^2 = 0$, that is, $y = 2$ and $3z = 2k\pi$. So the solutions are

$$x = 12, \quad y = 2, \quad z = \frac{2}{3}k\pi \quad (k \in \mathbb{Z})$$