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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. How many vertices can a simple complete graph have if the number of its edges is less than six times, but more than five times the number of its vertices?

(11 points)

Solution: In a simple complete graph every vertex is connected to every other vertex exactly once, so a simple complete graph with n vertices has $\frac{n(n-1)}{2}$ edges. By the conditions,

$$5n < \frac{n(n-1)}{2} < 6n$$

Since $n \geq 1$, this is $10 < n-1 < 12$, so $n-1 = 11$, $n = 12$.

2. Solve the following system of equations over the pairs of real numbers.

$$\begin{aligned} 4^{\frac{x+y}{1004}} - 4^{\frac{x+y}{2008}} &= 12 \\ 2007^{\log_x(2008y-x)} &= 1 \end{aligned}$$

(12 points)

Solution: The first equation can be written as

$$\left(4^{\frac{x+y}{2008}}\right)^2 - 4^{\frac{x+y}{2008}} = 12$$

which is quadratic in $4^{\frac{x+y}{2008}}$, with roots -3 and 4 . $4^{\frac{x+y}{2008}} = -3$ is impossible, and $4^{\frac{x+y}{2008}} = 4$ gives $x+y = 2008$. From the second equation, $\log_x(2008y-x) = 0$, that is, $2008y-x = 1$ (with $x > 0$, $x \neq 1$). From the two equations, $x = 2007$ and $y = 1$, and this is indeed a solution of the system.

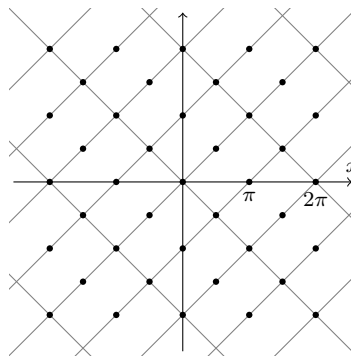
3. Find the points (x, y) of the coordinate plane whose coordinates satisfy

$$\cos(x + y) = 1$$

$$\sin(x - y) = 0$$

(14 points)

Solution: The first equation holds if and only if $x + y = 2k_1\pi$ ($k_1 \in \mathbb{Z}$); these points are on the lines $y = -x + 2k_1\pi$. The second equation holds if and only if $x - y = k_2\pi$ ($k_2 \in \mathbb{Z}$); these points are on the lines $y = x - k_2\pi$.



The points we are looking for are the intersection points of these lines:

$$\left((2k_1 + k_2) \frac{\pi}{2}, (2k_1 - k_2) \frac{\pi}{2} \right), \quad \text{where } k_1, k_2 \in \mathbb{Z}$$

(that is, the points $\left(a \frac{\pi}{2}, b \frac{\pi}{2} \right)$ where a and b are integers and $a + b$ is even).

4. A 24-person guesthouse has one 4-bed room, one 8-bed room, and one 12-bed room. In the 4-bed room the price is 1000 forints per person per night, in the 8-bed room 800 forints, and in the 12-bed room 700 forints. To save money, the rooms are heated depending on the number of guests, as shown in the table:

number of people	1–4	5–8	9–12	13–16	17–20	21–24
heated rooms	4-bed	8-bed	12-bed	4-bed + 12-bed	8-bed + 12-bed	all three

If more than one room is heated, the cheaper places can be filled first. A group of any size from 1 to 24 people can rent the house.

- For which group is the average price per person lower: a group of 15 or a group of 21 people renting the guesthouse?
- Is there a group size for which it is worth renting the house for more people than there are in the group?
- Order the group sizes in decreasing order of the average price per person.

(14 points)

Solution: a) For 15 people, 12 sleep for 700 forints and 3 for 1000 forints; the price per person is

$$\frac{12 \cdot 700 + 3 \cdot 1000}{15} = 760 \text{ forints}$$

For 21 people, 12 sleep for 700, 8 for 800, and one for 1000 forints; the price per person is

$$\frac{12 \cdot 700 + 8 \cdot 800 + 1 \cdot 1000}{21} \approx 752.4 \text{ forints}$$

So the average price per person is lower for 21 people than for 15.

b) This can happen when a cheaper room is opened because of the extra people.

If 4 people pay for 5: $5 \cdot 800 = 4000 = 4 \cdot 1000$, so the total price is the same. (3 or fewer people should not pay for 5.)

If 8 people pay for 9: $9 \cdot 700 = 6300 < 6400 = 8 \cdot 800$, so this is worth it. (7 or fewer people should not pay for 9.)

If 16 people pay for 17: $12 \cdot 700 + 5 \cdot 800 = 12400 = 12 \cdot 700 + 4 \cdot 1000$, so the total price is the same. (15 or fewer people should not pay for 17.)

c) The following table shows the total and the average price for each group size:

people	1	2	3	4	5	6	7	8
total	1000	2000	3000	4000	4000	4800	5600	6400
average	1000	1000	1000	1000	800	800	800	800
people	9	10	11	12	13	14	15	16
total	6300	7000	7700	8400	9400	10400	11400	12400
average	700	700	700	700	723.1	742.9	760	775
people	17	18	19	20	21	22	23	24
total	12400	13200	14000	14800	15800	16800	17800	18800
average	729.4	733.3	736.8	740	752.4	763.6	773.9	783.3

In decreasing order of the average price (group sizes in parentheses have the same average):

(1, 2, 3, 4), (5, 6, 7, 8), 24, 16, 23, 22, 15, 21, 14, 20, 19, 18, 17, 13, (9, 10, 11, 12)

Part II

5. A bag contains blue and red balls. The probability of drawing two blue balls in two draws is one fifth of the probability of drawing two red balls, and one sixth of the probability of drawing one ball of each color. Each ball is equally likely to be drawn. How many red and how many blue balls are in the bag?

(16 points)

Solution: Let p be the number of red balls and k the number of blue balls (the draws are without replacement). Then

$$\begin{aligned}
 P(2 \text{ blue}) &= \frac{k}{k+p} \cdot \frac{k-1}{k+p-1} = \frac{k(k-1)}{(p+k)(p+k-1)} \\
 P(2 \text{ red}) &= \frac{p(p-1)}{(p+k)(p+k-1)} \\
 P(\text{one of each}) &= \frac{p}{k+p} \cdot \frac{k}{k+p-1} + \frac{k}{k+p} \cdot \frac{p}{k+p-1} = \frac{2pk}{(p+k)(p+k-1)}
 \end{aligned}$$

From $6 \cdot P(2 \text{ blue}) = P(\text{one of each})$: $6k(k-1) = 2pk$, so $p = 3(k-1)$ (since $k \neq 0$). From $5 \cdot P(2 \text{ blue}) = P(2 \text{ red})$: $5k(k-1) = p(p-1)$. Substituting $p = 3k-3$ and rearranging: $k^2 - 4k + 3 = 0$, so $k = 1$ or $k = 3$. For $k = 1$ there would be no red balls, which is impossible. So there are 3 blue and 6 red balls in the bag.

6. The differences of the corresponding terms of a geometric sequence and an arithmetic sequence are 0, -3 , 10, 103. Find the two sequences.

(16 points)

Solution: The difference of the first terms is 0, so let the two sequences be a, aq, aq^2, aq^3 and $a, a + d, a + 2d, a + 3d$. From the differences of the corresponding terms:

$$\begin{aligned} (1) \quad a(q-1) - d &= -3 \\ (2) \quad a(q^2-1) - 2d &= 10 \\ (3) \quad a(q^3-1) - 3d &= 103 \end{aligned}$$

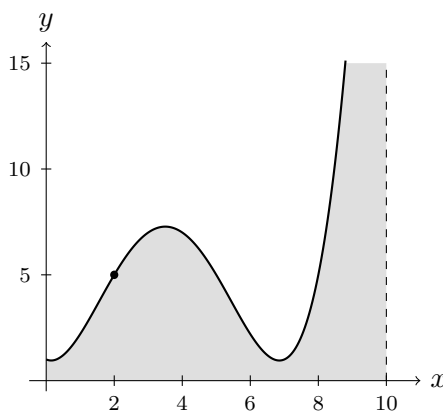
Subtracting twice (1) from (2): $a(q^2 - 2q + 1) = 16$, that is, $a(q-1)^2 = 16$. Subtracting (1) and (2) from (3): $a(q-1)(q^2 + q + 1 - q - 2) = 96$, that is, $a(q-1)^2(q+1) = 96$. Neither a nor $q-1$ is 0, so dividing, $q+1 = 6$, $q = 5$. Then $a = 1$ and $d = 7$. The two sequences are 1, 5, 25, 125, ... and 1, 8, 15, 22, ...

7. Find the equation of the line e through the point $(2, 5)$ of the function

$$f(x) = \frac{1}{20}(x+1)(x-2)(x-5)(x-8) + 5$$

such that on the interval $[0, 10]$ half of the area enclosed by the two curves is above the line e , and half is below it.

(16 points)



Solution: If the total area of the regions where the graph of f is above e equals the total area of the regions where it is below e , then the area under the graph of f on $[0, 10]$ equals the area under e :

$$\int_0^{10} f(x) dx = \int_0^{10} e(x) dx$$

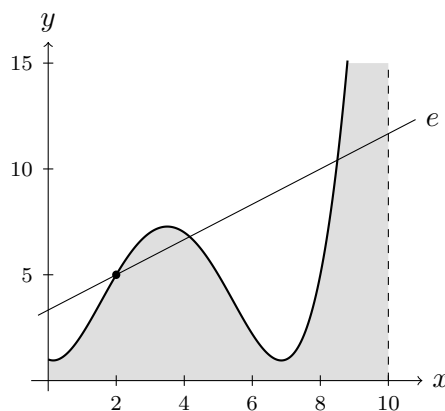
Let $e(x) = mx + b$. Expanding,

$$f(x) = \frac{x^4}{20} - \frac{7x^3}{10} + \frac{51x^2}{20} - \frac{7x}{10} + 1$$

so

$$\left[\frac{x^5}{100} - \frac{7x^4}{40} + \frac{17x^3}{20} - \frac{7x^2}{20} + x \right]_0^{10} = \left[m\frac{x^2}{2} + bx \right]_0^{10}, \quad \text{that is,} \quad 75 = 50m + 10b$$

The line passes through $(2, 5)$, so $2m + b = 5$. From the two equations, $m = \frac{5}{6}$ and $b = \frac{10}{3}$: the line is $e(x) = \frac{5}{6}x + \frac{10}{3}$.



8. From a group of people, k people can be chosen in p ways, $k+1$ people in $2p$ ways, and $k+2$ people in $3p$ ways. How many people are in the group?

(16 points)

Solution: Let the group have n people. Then

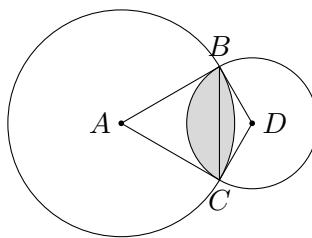
$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = p, \quad \binom{n}{k+1} = \frac{n!}{(k+1)!(n-k-1)!} = 2p, \quad \binom{n}{k+2} = \frac{n!}{(k+2)!(n-k-2)!} = 3p$$

From the first two equations, $\frac{k+1}{n-k} = \frac{1}{2}$, that is, $n = 3k + 2$. From the second and third, $\frac{k+2}{n-k-1} = \frac{2}{3}$, that is, $n = \frac{5k+8}{2}$. So $3k+2 = \frac{5k+8}{2}$, which gives $k = 4$ and $n = 14$. (Indeed, $\binom{14}{4} = 1001$, $\binom{14}{5} = 2002$, $\binom{14}{6} = 3003$.) There are 14 people in the group.

9. The shadow of the Earth falls on the disk of the Moon so that the common chord of the two circles is seen at an angle of 120° from the center of the Moon's disk, and at an angle of 60° from the center of the Earth's shadow. What fraction of the Moon's disk is not visible?

(16 points)

Solution: Let A be the center of the Earth's shadow, D the center of the Moon, and B, C the endpoints of the common chord. Let $AB = BC = CA = 1$ (the triangle ABC is equilateral, since the angle at A is 60°), and $DB = DC = x$.



By the Law of Cosines in the triangle BCD :

$$\cos \angle CDB = \frac{2x^2 - 1}{2x^2} = \cos 120^\circ = -\frac{1}{2}$$

so $x = \frac{1}{\sqrt{3}}$. The area of the Moon's disk is $\frac{\pi}{3}$. The common part consists of two circular segments: the sector ABC minus the triangle ABC gives $\frac{\pi}{6} - \frac{\sqrt{3}}{4}$, and the sector DBC minus the triangle DBC gives $\frac{\pi}{9} - \frac{\sqrt{3}}{12}$. So the common part has area $\frac{5\pi}{18} - \frac{\sqrt{3}}{3}$, and the fraction of the Moon's disk that is not visible is

$$\frac{\frac{5\pi}{18} - \frac{\sqrt{3}}{3}}{\frac{\pi}{3}} = \frac{5}{6} - \frac{\sqrt{3}}{\pi} \approx 0.28$$