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translated from: KoMaL, November 2008, pages 468-469

Pdf file: KoMaL, November 2008, pages 468-469

From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. Solve the following equation over the real numbers.

$$(x-2)^2 + \sqrt{4-4x+x^2} + \frac{6x-18}{3-x} = 0$$

(11 points)

Solution: The domain is $x \neq 3$. Equivalent transformations of the equation:

$$\begin{aligned} (x-2)^2 + \sqrt{(x-2)^2} + \frac{-6(3-x)}{3-x} &= 0 \\ (x-2)^2 + |x-2| - 6 &= 0 \\ (|x-2|+3)(|x-2|-2) &= 0 \end{aligned}$$

The first factor is positive for every x , and the second factor is 0 when $|x-2| = 2$. The two solutions of the equation are $x = 0$ and $x = 4$; both can be checked.

2. One of the most precious, strictly protected rare plants of the Aggtelek National Park is the Tornal wallflower (*Onosma tornensis*). Its notional value is 250 000 forints.

A researcher studying the plant found that the growth of the wallflower is approximately described by the function $h(x) = 2 + \log_2(x+1)$, where h is the height of the plant in cm, and x is the time elapsed in days, $x \in [0, 7]$.

- a) How tall is the plant on the last day of the measurement? (3 points)
 b) On which day does its height reach 4 cm? (3 points)
 c) The population is still significant in Hungary. About 30% of the population in the European Union is found here, about 600 000 plants. With a decrease of 4% per year, how long does it take for the Hungarian population to decrease to one quarter? (6 points)

Solution: a) $h(7) = 2 + \log_2 8 = 2 + 3 = 5$. On the last day of the measurement the plant is 5 cm tall.

b) The solution of $4 = 2 + \log_2(x+1)$ is $x = 3$. The height reaches 4 cm on the third day.

c) With a decrease of 4% per year, after n years the population is $600\,000 \cdot 0.96^n$, and this should equal one quarter of the population:

$$600\,000 \cdot 0.96^n = \frac{600\,000}{4}, \quad \text{that is,} \quad 0.96^n = 0.25, \quad n = \frac{\lg 0.25}{\lg 0.96} \approx 33.96$$

The population decreases to one quarter in about 34 years.

3. The hypotenuse of a right triangle is 5 units long, and the sines of its angles, in increasing order, are three consecutive terms of an arithmetic sequence. What is the area of the triangle?

(14 points)

Solution: In increasing order, the angles of the right triangle are α , $90^\circ - \alpha$, 90° . The sine function is strictly increasing from 0° to 90° , so $\sin \alpha < \cos \alpha < 1$. These are consecutive terms of an arithmetic sequence, so

$$\frac{1 + \sin \alpha}{2} = \cos \alpha$$

Squaring and using $\sin^2 \alpha + \cos^2 \alpha = 1$: $5 \sin^2 \alpha + 2 \sin \alpha - 3 = 0$, whose solutions are $\sin \alpha = 0.6$ and $\sin \alpha = -1$. -1 is impossible, since for an angle of a right triangle $0 < \sin \alpha < 1$. So the leg opposite α is $a = 5 \sin \alpha = 3$, and by the Pythagorean theorem the other leg is $b = 4$. The area of the triangle is

$$T = \frac{3 \cdot 4}{2} = 6$$

4. The two altitudes drawn from the vertex of one of the obtuse angles of a rhombus are 13 units long, and the segment joining their feet is 10 units long.

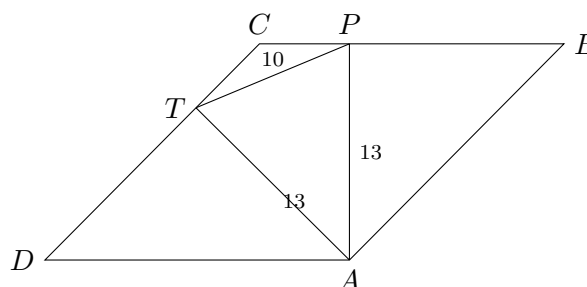
a) What are the angles of the rhombus?

(10 points)

b) What is the perimeter of the rhombus?

(4 points)

Solution: Let A be the obtuse vertex, and let T and P be the feet of the altitudes on the sides CD and BC , with $AT = AP = 13$ and $TP = 10$.



a) The angle between the two altitudes from A equals the acute angle α of the rhombus (the quadrilateral $ATCP$ has right angles at T and P , so $\angle TAP = 180^\circ - \angle C$, and $\angle C$ is the obtuse angle). The triangle ATP is isosceles, so the altitude to its base halves the angle at A :

$$\sin \frac{\alpha}{2} = \frac{5}{13}, \quad \text{so} \quad \frac{\alpha}{2} \approx 22.62^\circ, \quad \alpha \approx 45.24^\circ$$

The acute angles of the rhombus are about 45.24° , and the obtuse angles are about 134.76° .

b) In the right triangle APB (or ATD), $\sin 45.24^\circ = \frac{13}{a}$, so $a \approx 18.31$, and the perimeter of the rhombus is $K = 4a \approx 73.24$.

Part II

5. a) How many four-digit numbers can be formed from the even digits, if the digits cannot be repeated? (3 points)
 b) What is the sum of the four-digit numbers obtained in this way? (5 points)

c) We write the even digits on five slips, put them into an urn, draw four of them, and place them next to each other in the order of drawing. If the resulting number has four digits, what is the probability that it is divisible by six? (8 points)

Solution: a) We use the even digits 0, 2, 4, 6, 8. The first place can be 2, 4, 6, or 8; then the second place can also be filled in 4 ways, the third in 3 ways, and the fourth in 2 ways. So $4 \cdot 4 \cdot 3 \cdot 2 = 96$ such four-digit numbers can be formed.

b) The number of times each digit occurs in each place:

	0	2	4	6	8
thousands	0	24	24	24	24
hundreds	24	18	18	18	18
tens	24	18	18	18	18
units	24	18	18	18	18

So the sum is

$$(2 + 4 + 6 + 8) \cdot (24 \cdot 1000 + 18 \cdot 100 + 18 \cdot 10 + 18 \cdot 1) = 480\,000 + 36\,000 + 3600 + 360 = 519\,960$$

c) A number is divisible by 6 if it is divisible by both 2 and 3. The last digit is even, so we only need divisibility by 3, that is, the digit sum must be divisible by 3. The four digits used can be

I. 0, 2, 4, 6; II. 0, 2, 4, 8; III. 0, 2, 6, 8; IV. 0, 4, 6, 8; V. 2, 4, 6, 8

with digit sums 12, 14, 16, 18, 20. Only cases I and IV give numbers divisible by 3. In each of these cases, the first place can be filled in 3 ways, then 3, 2, 1, so the number of favorable cases is $2 \cdot (3 \cdot 3 \cdot 2 \cdot 1) = 36$. The number of all cases is 96, so the probability is

$$p = \frac{36}{96} = 0.375$$

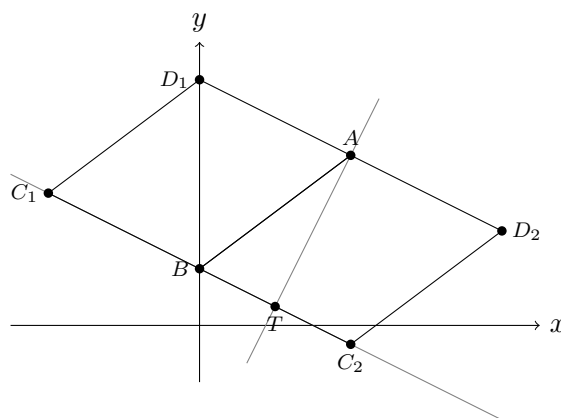
6. The coordinates of two adjacent vertices of a parallelogram are $A(8, 9)$ and $B(0, 3)$, and the equation of the line of one of its sides is $x + 2y = 6$. The area of the parallelogram is 80. Find the coordinates of the missing vertices.

(16 points)

Solution: Substituting, A is not on the given line, but B is; so the given line contains the side BC . The area of the parallelogram is $T = a \cdot m$, where $a = BC$ and the height m is the distance of A from the given line. The line through A perpendicular to $x + 2y = 6$ is $2x - y = 7$; the two lines meet at $T(4, 1)$. So the height is $m = AT = \sqrt{80} \approx 8.94$, and

$$a = \frac{80}{\sqrt{80}} = \sqrt{80} \approx 8.94$$

For $C(c_1, c_2)$: $BC = \sqrt{c_1^2 + (c_2 - 3)^2} = \sqrt{80}$ and $c_1 + 2c_2 = 6$. Solving this system gives two points: $C_1(-8, 7)$ and $C_2(8, -1)$. The diagonals of a parallelogram bisect each other, so the midpoint of AC is also the midpoint of BD ; this gives $D = A + C - B$: $D_1(0, 13)$ and $D_2(16, 5)$.



So the missing vertices are $C(-8, 7)$, $D(0, 13)$, or $C(8, -1)$, $D(16, 5)$.

7. The function

$$f(x) = \frac{2x^2 - 9x - 11}{x^2 - 5x - 6}$$

is defined on the real numbers $\mathbb{R} \setminus \{-1, 6\}$.

a) Graph the function f .

(8 points)

b) Find the smallest and the largest value of f on the interval $[0, 5]$.

(2 points)

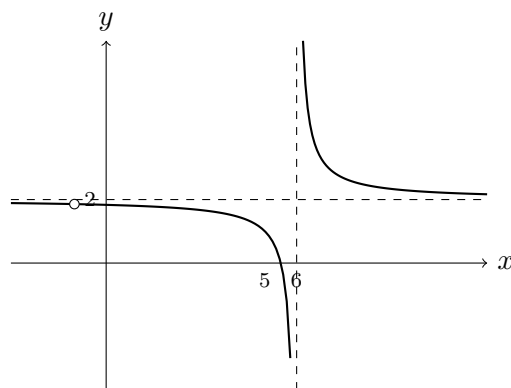
c) Find the equation of the tangent line of the graph of f at the point with x -coordinate 5.

(6 points)

Solution: a) In factored form,

$$\frac{2x^2 - 9x - 11}{x^2 - 5x - 6} = \frac{2(x - 5.5)(x + 1)}{(x - 6)(x + 1)}$$

and after simplifying, $f(x) = \frac{2x - 11}{x - 6} = \frac{1}{x - 6} + 2$ (with $x \neq -1$). The graph is a hyperbola with asymptotes $x = 6$ and $y = 2$, with the point at $x = -1$ removed.



b) The function is strictly decreasing on $[0, 5]$, so its smallest value is $f(5) = 1$, and its largest value is $f(0) = \frac{11}{6}$.

c) $f'(x) = -\frac{1}{(x - 6)^2}$, so the slope is $f'(5) = -1$, and the point of tangency is $(5, 1)$. The tangent line is $y - 1 = -(x - 5)$, that is, $y = -x + 6$.

8. We cut out of paper a square with side 4 dm and a regular triangle with side 4 dm, and then we cut the square along its diagonals into four congruent right triangles. From three isosceles right triangles and the equilateral triangle we can assemble the net of a pyramid, which is a 1 : 20 scale model of a building (considering the edges).

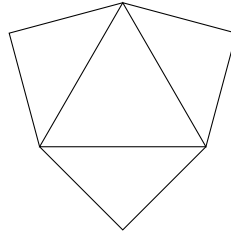
a) What is the surface area of the building, including the base? (9 points)

b) We want to make the solid building out of concrete. How much of each material do we use, if 10 m³ of concrete needs 2500 kg of cement, 12 m³ of gravel, and 2400 liters of water? (5 points)

c) What is the material cost of the concrete used, if from ordering to delivery (and payment) the material prices rose by 15%, and at the time of planning 1 quintal (100 kg) of cement cost 3100 forints and 1 m³ of gravel cost 4200 forints? (The water was taken from our own well, so it did not increase the costs.) (3 points)

Solution: a) By the similarity, the edges of the building are 20 times longer: 4 dm becomes 8 m. The three right triangles meet at the apex with right angles (like the corner of a cube), and their legs are $a = \frac{8}{\sqrt{2}} = 4\sqrt{2}$ m; the base is an equilateral triangle with side 8 m. The surface area is

$$A = \frac{8^2\sqrt{3}}{4} + 3 \cdot \frac{a^2}{2} = 16\sqrt{3} + 3 \cdot 16 \approx 75.71 \text{ m}^2$$



Note: KoMaL computes the area of the base as $8\sqrt{3}$, using the leg $4\sqrt{2}$ instead of the side 8 of the equilateral triangle, and gets 61.86 m^2 ; the correct value is about 75.71 m^2 .

b) Taking one of the right triangles as the base, the height is the third leg a , so the volume is

$$V = \frac{T \cdot a}{3} = \frac{16 \cdot 4\sqrt{2}}{3} \approx 30.17 \text{ m}^3$$

The ingredients for 10 m³ of concrete are known, so each amount must be multiplied by 3.017: about 7542.5 kg of cement, 36.204 m³ of gravel, and 7240.8 liters of water.

c) At the time of planning, the cement cost $75.425 \cdot 3100 = 233\,817.5$ forints and the gravel $36.204 \cdot 4200 = 152\,056.8$ forints, 385 874.3 forints in all. The material cost of the concrete is $385\,874.3 \cdot 1.15 \approx 443\,755$ forints.

9. The quadratic function $f(x) = x^2 + 2x + p^3 + 3p^2 + 2p$ has two different zeros. Find the value of the parameter p for which the product of the zeros equals the square of the value of f at 0. (16 points)

Solution: f has two different zeros, so the discriminant of $x^2 + 2x + p^3 + 3p^2 + 2p = 0$ is positive:

$$D = 4 - 4(p^3 + 3p^2 + 2p) > 0, \quad \text{that is,} \quad p^3 + 3p^2 + 2p < 1$$

By Viète's formulas, the product of the zeros is $x_1 x_2 = p^3 + 3p^2 + 2p$, and $f(0) = p^3 + 3p^2 + 2p$. So the condition is $p^3 + 3p^2 + 2p = (p^3 + 3p^2 + 2p)^2$; with $y = p^3 + 3p^2 + 2p$ this is $y = y^2$, so $y = 0$ or $y = 1$.

If $p^3 + 3p^2 + 2p = 0$: $p(p^2 + 3p + 2) = p(p+1)(p+2) = 0$, so $p = 0$, $p = -1$, or $p = -2$.

If $p^3 + 3p^2 + 2p = 1$: this contradicts the condition from the discriminant.

So the possible values of the parameter are $p = 0$, $p = -1$, $p = -2$ (then $f(x) = x^2 + 2x$, with zeros 0 and -2).