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From 2005, the Hungarian school-leaving exam (érettségi) in mathematics can be taken at two levels: standard level and advanced level. The advanced-level exam also serves as the university entrance exam in mathematics.

The written part of the advanced-level exam takes 240 minutes. Part I consists of 4 problems worth 51 points in all. Part II consists of 5 problems worth 16 points each, of which students choose and solve 4, for 64 points. So there is a total of 115 points possible.

This problem set is similar to such an exam. We advise the reader to work through the problem set while measuring the time. There are 240 minutes to solve and present the problems.

Part I

1. Solve the following equation over the real numbers.

$$\frac{\log_2(2x-5)}{\log_2(x^2-8)} = \frac{1}{2}$$

(12 points)

Solution: The conditions are $2x - 5 > 0$, $x^2 - 8 > 0$, and $\log_2(x^2 - 8) \neq 0$, that is, $x > 2\sqrt{2}$ and $x \neq 3$. We rearrange the equation with equivalent steps (using that the base 2 logarithm function is one-to-one):

$$2 \log_2(2x - 5) = \log_2(x^2 - 8)$$

$$(2x - 5)^2 = x^2 - 8$$

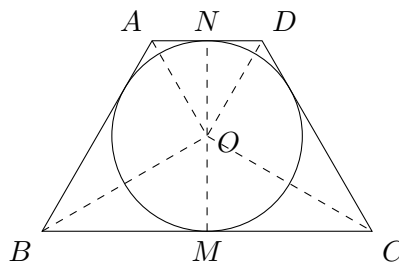
$$3x^2 - 20x + 33 = 0$$

The roots are $x_1 = 3$ and $x_2 = \frac{11}{3}$; the first is excluded by the conditions, and the second satisfies the equation. So the only solution is $x = \frac{11}{3}$.

2. What is the area of the isosceles trapezoid circumscribed about a circle of radius 3 cm, whose acute angles are 60° ?

(12 points)

Solution: Let the trapezoid be $ABCD$ with bases BC (longer) and AD , let O be the center of the circle, and let M and N be the midpoints of the bases.



The angles at B and C are 60° , so the angles at A and D are 120° . The center O of the inscribed circle is the intersection of the angle bisectors, so $\angle OBM = 30^\circ$ and $\angle OAN = 60^\circ$. In the right triangle BOM , $OM = 3$ is the radius, so $BO = 6$ and $BM = 3\sqrt{3}$; thus $BC = 6\sqrt{3}$. In the same way, from the right triangle ANO : $AN = \sqrt{3}$, $AD = 2\sqrt{3}$. The height of the trapezoid is $MN = 6$, so

$$T = \frac{(BC + AD) \cdot MN}{2} = \frac{(6\sqrt{3} + 2\sqrt{3}) \cdot 6}{2} = 24\sqrt{3} \approx 41.57 \text{ cm}^2$$

3. During parachutists' target jumps, it was recorded at how many meters from the center Berci landed in nine of his ten jumps. The recorded data in meters: 1, 3, 5, 2, 4, 6, 8, 9, 7. The mean absolute deviation of the data of his ten jumps from 5 m was 2.4 m.
- Find the missing tenth data value.
 - Find the mean, the mode, the median, and the standard deviation of the data of the ten jumps.

(13 points)

Solution: a) Let x be the tenth data value in meters. The sum of the absolute deviations from 5 is

$$|5 - 1| + |5 - 3| + |5 - 5| + \dots + |5 - 7| + |5 - x| = 20 + |5 - x|$$

The mean absolute deviation is 2.4, so $\frac{20 + |5 - x|}{10} = 2.4$, that is, $|5 - x| = 4$. So $x = 1$ or $x = 9$: the tenth data value can be 1 m or 9 m.

b) There are two possible data sets.

If the data are 1, 1, 2, 3, 4, 5, 6, 7, 8, 9: the mean is 4.6, the mode is 1, the median is 4.5, and the standard deviation is about 2.73.

If the data are 1, 2, 3, 4, 5, 6, 7, 8, 9, 9: the mean is 5.4, the mode is 9, the median is 5.5, and the standard deviation is about 2.73.

4. How many zeros are at the end of the number $\binom{100}{50}$?

(14 points)

Solution:

$$\binom{100}{50} = \frac{100!}{50! \cdot 50!}$$

In $100!$ the exponent of 5 is $20 + 4 = 24$, and in $50!$ it is $10 + 2 = 12$. So $100! = 5^{24} \cdot m$ and $50! = 5^{12} \cdot n$, where m and n are not divisible by 5, and

$$\binom{100}{50} = \frac{5^{24} \cdot m}{5^{12} \cdot n \cdot 5^{12} \cdot n} = \frac{m}{n^2}$$

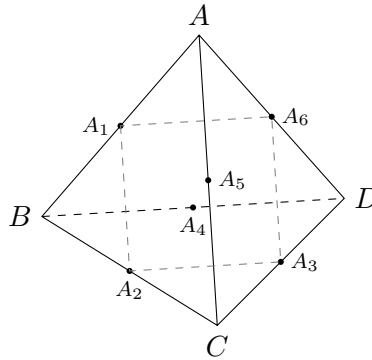
We know that $\binom{100}{50}$ is an integer, and $\frac{m}{n^2}$ is not divisible by 5. So there are no zeros at the end of $\binom{100}{50}$.

Part II

5. The opposite edges of a tetrahedron are perpendicular to each other. Show that there is a sphere that passes through the midpoints of all six edges.

(16 points)

Solution: Let the tetrahedron be $ABCD$, and let A_1, A_2, A_3, A_6 be the midpoints of the edges AB, BC, CD, DA , and A_4, A_5 the midpoints of BD and AC .



A_1A_2 and A_6A_3 are midlines of the triangles BAC and DAC , so both are parallel to AC and have length $\frac{1}{2}AC$. So the quadrilateral $A_1A_2A_3A_6$ has a pair of opposite sides that are parallel and equal: it is a parallelogram. Its sides A_1A_2 and A_2A_3 are parallel to AC and BD (A_2A_3 is a midline of the triangle CBD), and $AC \perp BD$, so $A_1A_2A_3A_6$ is a rectangle. The diagonals of a rectangle are equal and bisect each other, so with O the midpoint of its diagonals, $OA_1 = OA_2 = OA_3 = OA_6$.

In the same way, $A_1A_4A_3A_5$ is a rectangle (its sides are parallel to AD and BC , and $AD \perp BC$). The midpoint of its diagonal A_1A_3 is also O , so O is the center of this rectangle too: $OA_1 = OA_4 = OA_3 = OA_5$.

So O is the same distance from the midpoints of all six edges, and the sphere with center O and radius OA_1 passes through all of them.

6. Emese's monthly net salary is 150 000 forints. Assume that her net monthly salary is raised by 10% every year. After how many years can she buy an apartment worth 15 000 000 forints from her salary, if she saves 60% of her salary every month and the price of the apartment does not change meanwhile?

(16 points)

Solution: After one year her savings are $12 \cdot 150\,000 \cdot 0.6$. In the second year she saves $12 \cdot 150\,000 \cdot 1.1 \cdot 0.6$, in the third year $12 \cdot 150\,000 \cdot 1.1^2 \cdot 0.6$, and so on. So after N years her savings are

$$12 \cdot 150\,000 \cdot 0.6 \cdot [1 + 1.1 + 1.1^2 + \dots + 1.1^{N-1}] = 1\,080\,000 \cdot \frac{1.1^N - 1}{1.1 - 1} = 10\,800\,000 (1.1^N - 1)$$

We look for the smallest natural number N for which this reaches 15 000 000:

$$10\,800\,000 (1.1^N - 1) \geq 15\,000\,000, \quad \text{that is,} \quad 1.1^N \geq 2.3889$$

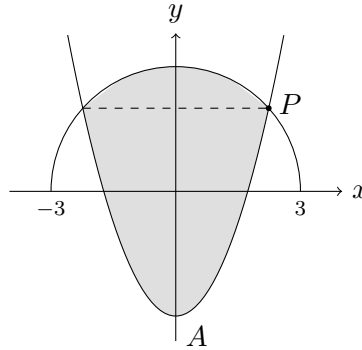
The smallest such N is 10, so she can buy the apartment after 10 years.

Note: More precisely, after 9 years she has about 14 666 000 forints, and in the 10th year she saves about 212 000 forints a month, so she can buy the apartment after 9 years and 2 months.

7. Two curves are given by the equations $y = \sqrt{9 - x^2}$ and $y = x^2 - 3$. Find the volume of the solid obtained by rotating the region bounded by the two curves about the y -axis.

(16 points)

Solution: The first equation is the upper semicircle with center at the origin and radius 3, and the second is a parabola with axis of symmetry the y -axis and vertex $A(0, -3)$.



We find the intersection point P : with $x^2 = y + 3$, we get $y = \sqrt{9 - (y + 3)} = \sqrt{6 - y}$ ($0 \leq y \leq 6$), so $y^2 + y - 6 = 0$, $y = 2$, and $P(\sqrt{5}, 2)$.

We integrate with respect to y . For $-3 \leq y \leq 2$ the radius of the cross section is $\sqrt{y + 3}$ (from the parabola), and for $2 \leq y \leq 3$ it is $\sqrt{9 - y^2}$ (from the circle):

$$V = \pi \int_{-3}^2 (y + 3) dy + \pi \int_2^3 (9 - y^2) dy = \pi \left[\frac{y^2}{2} + 3y \right]_{-3}^2 + \pi \left[9y - \frac{y^3}{3} \right]_2^3 = \frac{25}{2}\pi + \frac{8}{3}\pi = \frac{91}{6}\pi \approx 47.6$$

Note: KoMaL gives $\frac{32}{3}\pi$, which is an arithmetic slip in evaluating the integrals; the value $\frac{91}{6}\pi$ is confirmed by the shell method, $2\pi \int_0^{\sqrt{5}} x [\sqrt{9 - x^2} - (x^2 - 3)] dx$.

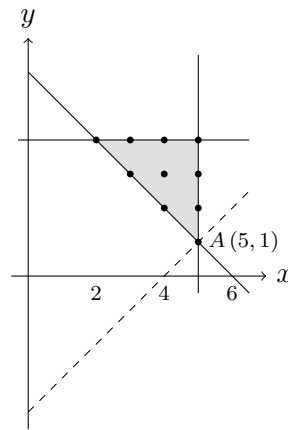
8. Three companies deliver concrete to a construction site. The first has 5, the second 4, and the third 6 concrete trucks. On a given day the construction needs 12 truckloads of concrete. How many trucks should be ordered from each company so that the delivery cost is minimal, if the delivery cost per truck is 40 000, 60 000, and 50 000 forints for the three companies?

(16 points)

Solution: Let x, y, z be the numbers of trucks ordered from the first, second, and third companies. Then $x + y + z = 12$, $0 \leq x \leq 5$, $0 \leq y \leq 4$, $0 \leq z \leq 6$, and we minimize $f = 40\,000x + 60\,000y + 50\,000z$. From the first equation, $z = 12 - x - y \leq 6$, that is, $x + y \geq 6$. Substituting z :

$$f = -10\,000x + 10\,000y + 600\,000, \quad \text{that is,} \quad y = x + \frac{f}{10\,000} - 60$$

The inequalities $x + y \geq 6$, $0 \leq x \leq 5$, $0 \leq y \leq 4$ give the triangle ABC in the figure. The line $y = x + \frac{f}{10\,000} - 60$ is parallel to the line $y = x$, and its y -intercept $\frac{f}{10\,000} - 60$ is minimal when it passes through the “lowest” lattice point of the triangle, $A(5, 1)$.



Substituting: $1 = 5 + \frac{f}{10\,000} - 60$, that is, $f = 560\,000$. So the best order is 5 trucks from the first company, 1 from the second, and 6 from the third; the minimal delivery cost is 560 000 forints. (We get the same result by choosing the cheapest available trucks first.)

9. The quadratic equation $ax^2 + bx + c = 0$ has real roots. We know that $|a + b + c| < |a|$. Prove that at least one root of the quadratic equation is in the interval $(0, 2)$.

(16 points)

Solution: Dividing the inequality by $|a|$:

$$\left| 1 + \frac{b}{a} + \frac{c}{a} \right| < 1, \quad \text{that is,} \quad \left| 1 - \left(-\frac{b}{a} \right) + \frac{c}{a} \right| < 1$$

By Viète's formulas, $x_1 + x_2 = -\frac{b}{a}$ and $x_1x_2 = \frac{c}{a}$, so $|1 - (x_1 + x_2) + x_1x_2| < 1$, that is,

$$|1 - x_1| \cdot |1 - x_2| < 1$$

This holds only if at least one of the two factors is less than 1. If, say, $|1 - x_1| < 1$, then $-1 < 1 - x_1 < 1$, that is, $0 < x_1 < 2$. So at least one root is indeed in the interval $(0, 2)$.